Probability Theory

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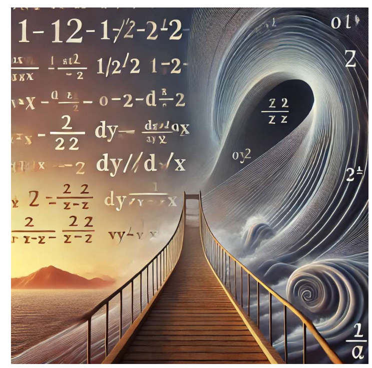
Previous Lecture

- Median/Mode
- Moments: Central/Standardized
- Skewness/Kurtosis
- Sample Moments/Law of Large Numbers

§6.4 - Moment Generating Functions (MGFs)



Generating functions are a bridge between sequences of numbers and the world of calculus. Start with a sequence of numbers, then create a continuous function – the generating function – that encodes the sequence. We then have all the tools of calculus at our disposal for manipulating the generating function.

A moment generating function encodes the moments of a distr.

Def (**Moment Generating Function**): If it exists, the MGF of X is $M(t) = E(e^{tX})$ as a function of t.

The MGF exists if $M(t)$ is finite on some open interval $(-a,a)$ containing 0. Otherwise the MGF of X doesn't exist.

The inclusion of t let's us use calculus!

$M(0) = 1$ for any valid MGF M. So, whenever you compute an MGF, plug in 0 and see if you get 1, as a quick check!

Ex (**Bernoulli MGF**): For $X \sim Bern(p)$, e^{tX} takes on the value e^t with prob p and the value 1 with prob q, so $M(t) = E(e^{tX}) = pe^t + q$.

Since this is finite for all values of t, the MGF is defined on $\mathbb{R}$.

Ex (**Geometric MGF**): For $X \sim Geom(p)$, the MFG is $M(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk}q^k p = p\sum_{k=0}^{\infty}(qe^t)^k = \frac{p}{1-qe^t}$, for $qe^t < 1$, i.e., for t in $(-1, log\frac{1}{q})$, which is an open interval containing 0.

Ex (**Uniform MGF**): For $U \sim Unif(a,b)$, the MGF is $M(t) = E(e^{tU}) = \frac{1}{b-a}\int_a^b e^{tu}du = \frac{e^{tb}-e^{ta}}{t(b-a)}$ for $t \neq 0$, and $M(0) = 1$.

MGFs:

- Encode a rv's moments.
- Determine a rv's distr, (like CDF and PF).
- Make it easy to find the distr of a sum of indep rvs.

Thm (**Moments via Derivatives of the MGF**): Given the MGF of X, we can get the nth moment of X by evaluating the nth derivative of the MGF at 0. So $E(X^n) = M^{(n)}(0)$.

Proof: Note that the Taylor expansion of $M(t)$ about 0 is $M(t) = \sum_{n=0}^{\infty} M^{(n)}(0)\frac{t^n}{n!}$.

On the other hand, we also have $M(t) = E(e^{tX}) = E\left(\sum_{n=0}^{\infty} X^n \frac{t^n}{n!}\right)$. (using taylor exp of e^{tX})

Next, we're allowed to interchange the position of the expectation and the infinite sum above because certain technical conditions are satisfied (this is where we invoke the condition that $E(e^{tX})$ is finite in an interval around 0). So $M(t) = \sum_{n=0}^{\infty} E(X^n)\frac{t^n}{n!}$.

Matching the cofficients of the two expansions, we get $E(X^n) = M^{(n)}(0)$. ■

So, w/the MGF, it is possible to find moments by taking derivatives rather than doing integrals!

Thm (**MGF Determines the Distr**). The MGF of a rv determines its distr. if two rvs have the same MGF, they have the same distr.

If there's even **a tiny interval** $(-a, a)$ containing 0 on which the MGFs are equal, the rvs must have the same distr.

[Proof Requires Analysis]

Thm (**MGF of a Sum of Indep Rvs**): If X and Y are indep, then the MGF of $X + Y$ is the product of the individual MGFs: $M_{X+Y}(t) = M_X(t)M_Y(t)$.

This is true because if X and Y are indep, then $E(e^{t(X+Y)}) = E(e^{tX})E(e^{tY})$ (this follows from results in Chapter 7).

Using this fact, we can get the MGFs of the Binomial and Negative Binomial, which are sums of indep Bernoullis and Geometrics, respectively.

Proposition (**MGF of Location-Scale Transformation**): If X has MGF $M_X(t)$, then the MGF of $bX + a$ is $E(e^{t(bX+a)}) = e^{at}E(e^{btX}) = e^{at}M_X(bt)$.

Ex (Moments via Derivatives of MGF): Let X have the PF: $f(x) := \begin{cases} \frac{2}{11} & \text{if } X = -1, \\ \frac{3}{11} & \text{if } X = 0, \\ \frac{4}{11} & \text{if } X = 1, \\ \frac{2}{11} & \text{if } X = 2, \\ 0 & \text{otherwise.} \end{cases}$

a. Find $M_X(t)$.

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=-1}^{2} e^{tx} f(x) = \tfrac{2}{11}e^{-1t} + \tfrac{3}{11}e^{0t} + \tfrac{4}{11}e^{t} + \tfrac{2}{11}e^{2t}$$

$$= \tfrac{3}{11} + \tfrac{2}{11}e^{-t} + \tfrac{4}{11}e^{t} + \tfrac{2}{11}e^{2t}.$$

b. Using $M_X(t)$, find $E(X)$.

$$M_X'(t) = -\tfrac{2}{11}e^{-t} + \tfrac{4}{11}e^{t} + \tfrac{4}{11}e^{2t}.$$

$$M_X'(0) = -\tfrac{2}{11}e^{0} + \tfrac{4}{11}e^{0} + \tfrac{4}{11}e^{2\cdot 0} = \tfrac{6}{11} = E(X).$$

c. Let $Y = 7X - 5$. Use $M_X(t)$ to find $M_Y(t)$.

$$M_Y(t) = e^{-5t}M_X(7t)$$

$$= e^{-5t}\left(\tfrac{3}{11} + \tfrac{2}{11}e^{-7t} + \tfrac{4}{11}e^{7t} + \tfrac{2}{11}e^{14t}\right)$$

$$= \tfrac{3}{11}e^{-5t} + \tfrac{2}{11}e^{-12t} + \tfrac{4}{11}e^{2t} + \tfrac{2}{11}e^{9t}.$$

Ex (Uniform MGF):

a. Find the MGF ($M_U(t)$) of $U \sim Unif(0,1)$.

$$M_U(t) = E(e^{tU})$$

$$= \tfrac{1}{1-0}\int_0^1 e^{tu}\,du = \tfrac{e^t-1}{t} \text{ for } t \neq 0.$$

b. Using M_X from the previous example, and assuming U and X are indep find $M_{U+X}(t)$.

$$M_{U+X}(t) = M_U(t)M_X(t) = \tfrac{e^t-1}{t}\left(\tfrac{3}{11} + \tfrac{2}{11}e^{-t} + \tfrac{4}{11}e^{t} + \tfrac{2}{11}e^{2t}\right).$$

c. Use $M_U(t)$ to find the mean and variance of U.

$$M_U^{'}(t) = \frac{te^t-(e^t-1)}{t^2} = \frac{(t-1)e^t+1}{t^2}.$$

$$M_U^{''}(t) = \frac{e^t}{t} - \frac{2(t-1)e^t+2}{t^3}.$$

$$E(X) = M_U^{'}(0) = \frac{(t-1)e^t+1}{t^2}\Big|_{t=0}$$

$$\overset{L'H}{=} \frac{e^t+(t-1)e^t}{2t}\Big|_{t=0}$$

$$\overset{L'H}{=} \frac{2e^t+(t-1)e^t}{2}\Big|_{t=0}$$

$$= \frac{2-1}{2} = \frac{1}{2}.$$

$$E(X^2) = M_U^{''}(t) = \left(\frac{e^t}{t} - \frac{2(e^t(t-1)+1)}{t^3}\right)\Big|_{t=0}$$

$$\overset{L'H}{=} \left(e^t - \frac{2}{3}\frac{e^t}{t}\right)\Big|_{t=0} = \left(\frac{\left(t-\frac{2}{3}\right)e^t}{t}\right)\Big|_{t=0}$$

$$\overset{L'H}{=} \left(e^t + \left(t-\frac{2}{3}\right)e^t\right)\Big|_{t=0}$$

$$= 1 - \frac{2}{3} = \frac{1}{3}.$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

Harvard Videos:

youtube.com/watch?v=N8O6zd6vTZ8&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=18

youtube.com/watch?v=tVDdx6xUOcs&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=19

youtube.com/watch?v=xiVWNkQUqKk&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=20

What did we learn?

- Moment Generating Functions (MGFs)
- Moments via Derivatives of MGFs