Probability Theory

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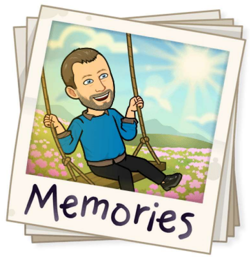
Previous Lecture

- LOTUS
- Variance/Standard Deviation (SD)



§4.7 - Poisson Distr

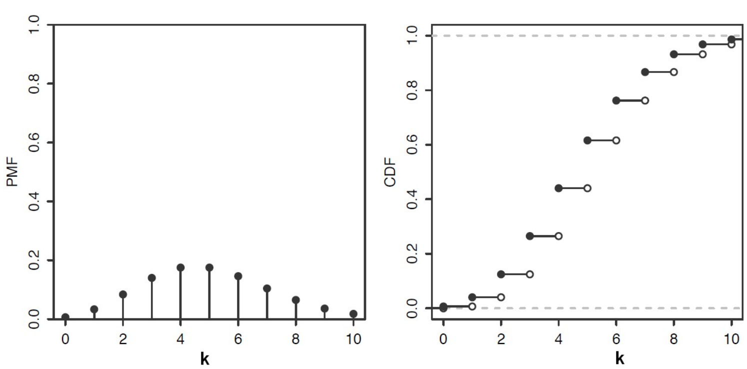
Poisson distr is used in situations where we're counting the # of successes in a particular region or interval of time, and there are a large # of trials, each with a small prob of success.

- # of emails you receive in an hour. 
- # of chips in a chocolate chip cookie. 
- # of earthquakes in a year in some region. 

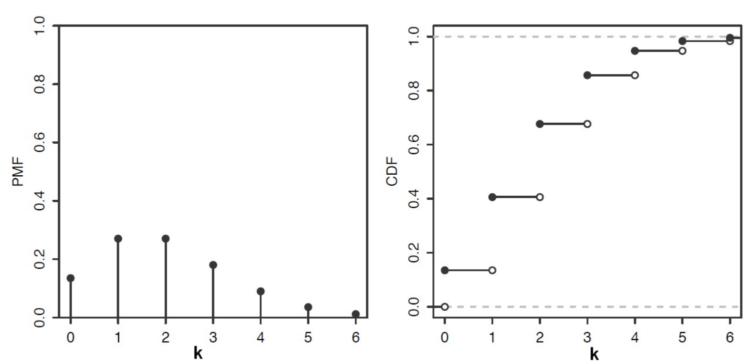
If we let λ be the rate of occurrence of these rare events, then a good approximation of the distr is:

Def (**Poisson Distr**): Discrete X has Poisson distr w/parameter λ, where $\lambda > 0$, if the PF of X is $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$ We write this as $X \sim Pois(\lambda)$.

This is a valid PMF because of the Taylor series $e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$. So, $\sum_{k=0}^{\infty} P(X = k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$.



$Pois(5)$ PF and CDF



$Pois(2)$ PF and CDF

Ex (**Poisson**): While sitting at the park, you notice dog walkers pass at a rate of 8 per hr.
Let X be the # of dog owners who walk by in the next hr. What is the prob that $X = 6$?

Solution: Observe that $\lambda = 8$. So, $P(X = 6) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-8}8^6}{6!} \approx 0.1221$.

Poisson Expectation and Variance

Let $X \sim Pois(\lambda)$. The mean and variance are both equal to λ.

Proof: For the mean, we have: $E(X) = e^{-\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}$

$= e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!}$ (the $k = 0$ term equals 0)

$= \lambda e^{-\lambda}\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$ (factored out a λ, and absorbed k into denom, now looks like Taylor series of e^{λ})

$= \lambda e^{-\lambda}e^{\lambda} = \lambda$.

For variance, you can check out the proof in the book.

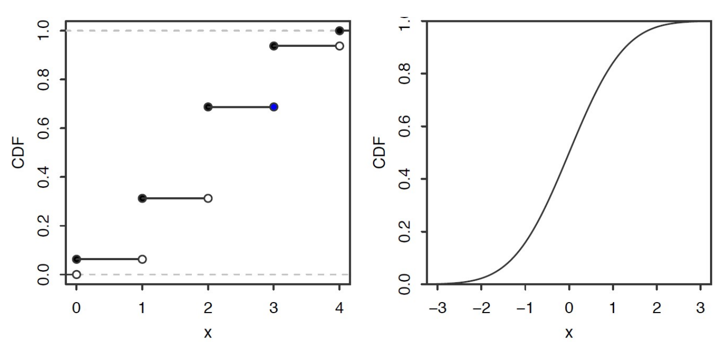
Activity 9

Chpt 5: Continuous Random Variables

Continuous rvs can take on any real value in an interval (possibly of infinite length, such as $(0, \infty)$ or the entire real line $\mathbb{R}$).

Def (**Continuous Rv**): A rv has a **continuous distr** if its CDF is differentiable.
We also allow there to be endpoints (or finitely many points) where the CDF is continuous but not differentiable, as long as the CDF is differentiable everywhere else.

A **continuous rv** is a rv with a continuous distr.



CDF: Discrete vs. Continuous Rv

§5.1 - Probability Density Functions (PDF/PF)

CDFs of discrete rvs are difficult to work with (can't differentiate). But for cont. rvs we have:

Def (**Probability Density Function**, **PDF/PF**): For X cont. w/CDF F, the PDF of X is the derivative f of the CDF, given by $f(x) = F'(x)$: The support of X, and of its distr, is the set of all x where $f(x) > 0$.

❗ For a cont. X, we have $P(X = x) = 0, \forall x$. This is because the CDF of X has not jumps!

A PMF of X is zero everywhere, so instead we work with PDFs.

Unlike a PMF, a PDF isn't a prob. To get a prob we must integrate the PDF.

Proposition (**PDF → CDF**): Let X be cont. w/PDF f. Then the CDF of X is: $F(x) = \int_{-\infty}^{x} f(t)dt$.

(like summing over the PMF of a discrete var)

Proof. By definition of PDF, F is an antiderivative of f. So by the fundamental thm of calculus: $\int_{-\infty}^{x} f(t)dt = F(x) - F(-\infty) = F(x)$. ■

For cont/discrete X, we will use PF to mean PDF/PMF resp.

Since the PF determines X's distr, we can use it to find the prob of X falling into an interval (a, b).

Ex (**Continuous Rv Prob**): Let cont. X have PF: $f(x) = \begin{cases} \frac{x^3}{156} & 1 \le x \le 5, \\ 0 & \text{otherwise.} \end{cases}$

Find $P(4 \le X \le 5)$.

$$P(4 \le X \le 5) = F(5) - F(4) \qquad \text{(FTC and CDF)}$$

$$= \tfrac{1}{156}\int_4^5 x^3 dx = \tfrac{1}{156}\left[\tfrac{1}{4}x^4\right]_4^5 = \tfrac{1}{4\cdot 156}(5^4 - 4^4) = \tfrac{123}{208}.$$

What's $P(1 \le X \le 5)$?

$P(1 \le X \le 5) = 1$, since X's support is restricted to $[1, 5]$. □

Thm (**Valid PFs**): The PF f of a cont. rv must satisfy: ...

- Nonnegative: $f(x) \ge 0$,
- Integrates to 1: $\int_{-\infty}^{\infty} f(x)dx = 1$.

Proof. The first criterion is true because prob is nonnegative.

(if $f(x_0)$ were negative, then we could integrate over a tiny region around x_0 and get a negative prob!).

Alternatively, note that the PF at x_0 is the slope of the CDF at x_0, so $f(x_0) < 0$ would imply the CDF is decreasing at x_0, which isn't allowed.

The second criterion is true since $\int_{-\infty}^{\infty} f(x)dx$ is the prob of X falling somewhere on $\mathbb{R}$, which is 1. ■

Expectation/Linearity/LOTUS

The definition of expectation for cont. rvs is analogous to that for discrete rvs: just replace Σ with $\int dx$.

Def (**Expectation of a Continuous Rv**): The *expected value* (also called *expectation* or *mean*) of a cont. X w/PF f is $E(X) = \int_{-\infty}^{\infty} xf(x)dx$.

LOTUS and Linearity of expectation also hold analogously for cont. rvs.

Thm (**LOTUS**, **Cont**): Let X be cont. w/PF f, and let $g(x) : \mathbb{R} \to \mathbb{R}$, then: $E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$.

Ex: Let cont. X have the PF: $f(x) = \begin{cases} \frac{x^3}{156} & 1 \le x \le 5, \\ 0 & \text{otherwise.} \end{cases}$

Find $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{156}\int_1^5 x^4dx = \frac{1}{156}\left[\frac{1}{5}x^5\right]_1^5 = \frac{1}{156 \cdot 5}(5^5 - 1) = \frac{781}{195}.$$

Find $E(X^2)$.

$$E(X^2) = \int_{-\infty}^{\infty} x^2f(x)dx = \frac{1}{156}\int_1^5 x^5dx = \frac{1}{156}\left[\frac{1}{6}x^6\right]_1^5 = \frac{1}{156 \cdot 6}(5^5 - 1) = \frac{781}{234}.$$

Activity 10

Harvard Video: youtube.com/watch?v=Tci—bVs60&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=13

What did we learn?

- Poisson
- Continuous Rv: PF/CDF
- Cont. Valid PFs
- Cont. Expectation/LOTUS