Probability Theory

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Previous Lecture

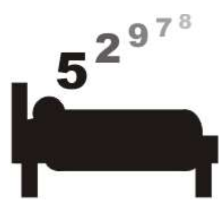
- Expectation of a Discrete Rv
- Geometric Distr/PF/CDF/Expectation
- Negative Binomial Distr/PF/Expectation

§4.5/4.6 - LOTUS/Variance

Given X, let $g(X) = \frac{\sqrt{X}}{X^2+3}$. We know how to calculate $E(X)$, but how do we calculate $E(g(X))$?

This would be annoying to calculate: $E(g(X)) = \Sigma_{g(x)} g(x) P(g(x) = x)$. Instead...

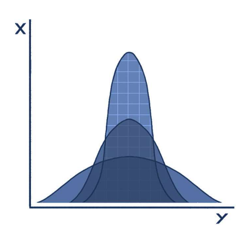
Thm (**Law of the Unconscious Statistician**, **LOTUS**): If X is discrete and $g(x) : \mathbb{R} \to \mathbb{R}$, then $E(g(X)) = \Sigma_x g(x) P(X = x)$, where the sum is taken over all possible values x of X.



In other words, we can calculate $E(g(X))$ by knowing $P(X = x)$, we need **not** know the PF of $g(X)$.

Variance/Std Dev

Measurements of the spread between numbers in a data set (e.g., a distr).



Def (**Variance**): The variance of X is $Var(X) = E\left((X - E(X))^2\right)$.

(average squared distance from the mean)

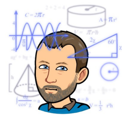
Variance is a single number summary of X (like expectation), measuring the spread of X's distr.

Expected value tells us the center-of-mass, and variance tells us spread.

Def (**Standard Deviation**, **SD**): The square root of the variance is **SD**: $SD(X) = \sqrt{Var(X)}$.

(average distance from the mean)

But how do we calculate the variance?



If we let $\mu = E(X)$. Then expanding $(X - \mu)^2$...

$$E\left((X-\mu)^2\right) = E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + \mu^2 \qquad \text{(linearity)}$$

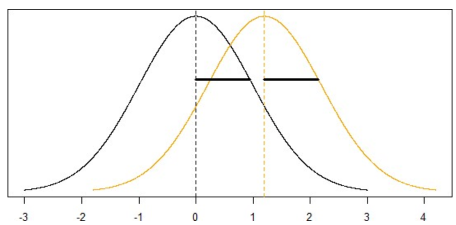
$$= E(X^2) - \mu^2$$

Thm (**Var from Expectation**): For any X, we have: $Var(X) = E\left((X-\mu)^2\right) = E(X^2) - (E(X))^2$.

Proof: See above. ■

Properties

- ♦ $Var(X + c) = Var(X)$ for any constant c.
- ♦ $Var(cX) = c^2 Var(X)$ for any constant c.



$Var(X + c) = Var(X)$

- ♦ If X and Y are indep, then $Var(X + Y) = Var(X) + Var(Y)$.

  (observe when $X \equiv Y$ that $Var(X + Y) = Var(2X) = 4Var(X) > 2Var(X) = Var(X) + Var(Y)$.)

- ♦ $Var(X) \geq 0$, with equality if and only if $P(X = a) = 1$ for some constant a. (Var is an average distance, so positive)

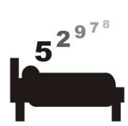
Ex: Let X have PF: $f(x) = \begin{cases} 0.1 & \text{for } x = -2, \\ 0.5 & \text{for } x = 1, \\ 0.25 & \text{for } x = 0, \\ 0.25 & \text{for } x = 2 \\ 0 & \text{otherwise.} \end{cases}$

Find $E(X)$.

$$E(X) = -2(0.1) + 1(0.5) + 0(0.25) + 2(0.25) = 0.8.$$

Let $Y = X^2$. Find $E(Y)$.

$$E(Y) = \Sigma_x X^2 P(X = x) \qquad \text{(LOTUS)}$$



$$= (-2)^2(0.1) + (1)^2(0.5) + (0)^2(0.25) + (2)^2(0.25) = 1.9$$

Find $E(3X - Y + 1)$

$$E(3X - Y + 1) = 3E(X) - E(Y) + E(1) \qquad \text{(linearity of expectation)}$$

$$= 3(0.8) - 1.9 + 1 = 1.5.$$

Find $Var(2X - 5)$.

$$Var(2X - 5) = 4Var(X) \qquad \text{(Var properties)}$$

$$= 4E(X^2) - 4E(X)^2 \qquad \text{(Var from Expectation Thm)}$$

$$= 4E(Y) - 4(0.8)^2 = 4(1.9) - 4(0.64) = 5.04$$

Activity 8

What did we learn?

- LOTUS
- Variance/Standard Deviation (SD)