Probability Theory

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Previous Lecture

- Random Vars (rvs)
- Discrete Rvs
- Prob Mass Functions (PFs)
- Valid PFs

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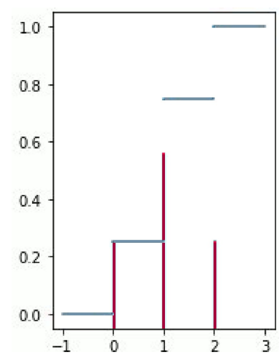
§3.6 - Cumulative Distribution Functions (CDFs)



PF in red, CDF in blue

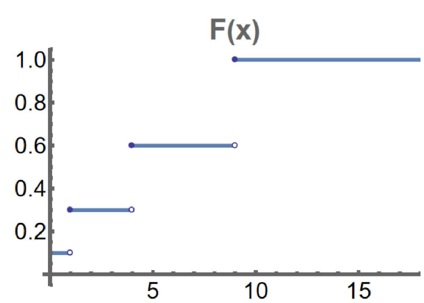
What's the prob that X is less than some value x?

Def (**Cumulative Distr Function**, **CDF**): A CDF of X, denoted F_X is given by $F_X(x) := P(X \le x)$. Or just $F(x) := P(X \le x)$.

Ex (**CDF**): Let X have the following PF: $f(x) = \begin{cases} \frac{\sqrt{x}}{10} & \text{for } x = 1, 4, 9, 16 \\ 0 & \text{otherwise.} \end{cases}$

Find the CDF $F(x)$ of X. And plot it.

$$F(x) = \begin{cases} 0.1 & x < 1 \\ 0.3 & 1 \le x < 4 \\ 0.6 & 4 \le x < 9 \\ 1 & 9 \le x < 16 \end{cases}$$



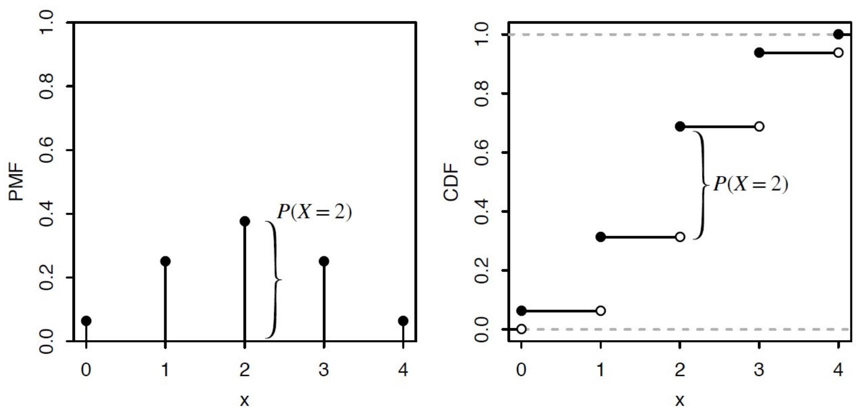
From PF → CDF: To find $P(X \le 4.5)$, which is the CDF evaluated at 4.5, we sum the PF over all values of support that are less than or equal to 4.5. So, $P(X \le 4.5) = P(X = 1) + P(X = 4) = \frac{\sqrt{1}}{10} + \frac{\sqrt{4}}{10} = \frac{3}{10}$.

More generally, the value of the CDF at an arbitrary point x (so, $P(X \le x)$) is the sum of the heights of the vertical bars of the PF at values less than or equal to x.

From CDF → PF: The height of a jump in the CDF at x is equal to the value of PF at x.

In the plot below, the height of the jump at 2 is same as height of corresponding vertical bar in the PF.
Flat regions of CDF correspond to values outside the support, so PF is equal to 0 there.

Ex:



$X \sim Bin(4, \frac{1}{2})$

Thm (**Valid CDFs**): Any CDF F is:

- **Increasing**: If $x_1 \le x_2$, then $F(x_1) \le F(x_2)$. (since probs are positive)

- **Right-Continuous**: As in above figure, CDF is continuous except possibly having some jumps.
  At jumps, the CDF is continuous from the right: for any a, we have: $F(a) = \lim_{x \to a^+} F(x)$.

- Convergence to 0 and 1 in the limits: $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

[Proofs in Book]

Harvard Video: youtube.com/watch?v=LX2q356N2rU&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=9

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§3.7 - Functions of Random Variables

What if we add two rvs $X + Y$?

Just add the result! How about X^2, or $\frac{\sqrt{X}}{\ln Y^2}$? And are the result rvs ?!?

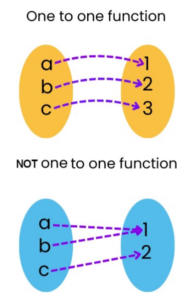
Yes! A function $f(X)$ of a rv X **IS** a rv. For example: X^2, e^X, $\sin(X)$, etc.

Def (**Function of a Rv**): For an experiment w/sample space S, a rv X, and $g : \mathbb{R} \to \mathbb{R}$, we define $Y := g(X)$. Y is a rv that maps s to $g(X(s))$ for all $s \in S$.

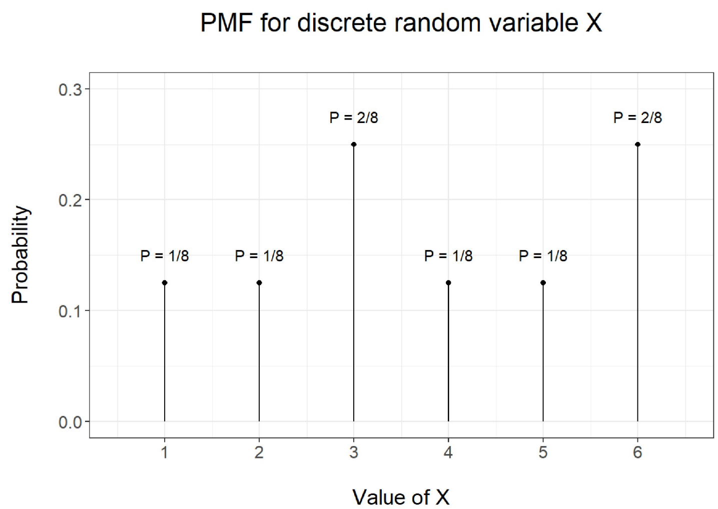
This is a composition of functions. We are saying, "first apply $X(s)$, then apply $g(x)$."

If we know the PF for X, can we find the PF for $Y = g(X)$? (yes, let's see how)

If g is a **one-to-one** function, the support of Y is the set of all $g(x)$ with x in the support of X.



The graph below depicts the PFs for two rvs. For the first, X has support $\{1,2,3\}$. For the second, $g(X) = X + 3$, has support $\{4,5,6\}$.



Ex (**One to One**): Define X w/PF: $f_X(x) = \begin{cases} \frac{x}{6} & \text{for } x = 1,2,3 \\ 0 & \text{otherwise.} \end{cases}$

Let $Y = 3X$ (one to one!), what is the PF of Y?

Solution: Notice our Y values are $\{3,6,9\}$.

So $P(Y = 3) = P(3X = 3) = P(X = 1) = \frac{1}{6}$,

$P(Y = 6) = P(X = 2) = \frac{2}{6}$, and $P(Y = 9) = P(X = 3) = \frac{3}{6}$.

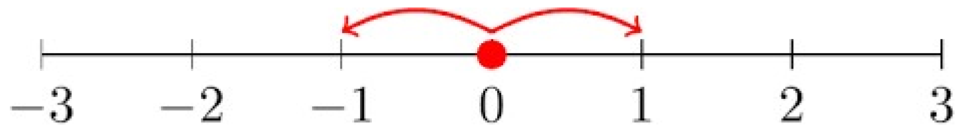
So, $f_Y(y) = \begin{cases} \frac{y}{3 \cdot 6} & \text{for } y = 3,6,9 \\ 0 & \text{otherwise.} \end{cases}$

Sanity check: $\frac{3}{18} + \frac{6}{18} + \frac{9}{18} = 1$. □

Strategy: To find the PF of a rv w/unfamiliar distr: express the rv as a one-to-one function of a known distr.

Ex (**Random Walk**). A particle moves n steps on a number line. It starts at 0, and at each step it moves 1 to the right or left, with equal prob's. Assume all steps are indep. Let Y be it's position after n steps.

Find the PF of Y.



Solution: Consider each step to be a Bernoulli trial, where right is a "success" and left a "failure."

The # of steps the particle takes to the right is a $Bin(n, \frac{1}{2})$ rv, which we name X.

If $X = j$, then we have j steps to the right and $n - j$ to the left, giving a final position: $j - (n - j) = 2j - n$.

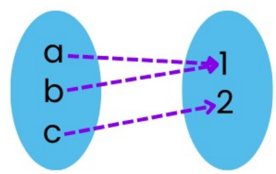
So we can express Y as a one-to-one function of X: $Y = 2X - n$.
Since X takes values in $\{0, 1, 2, \ldots, n\}$, Y takes values in $\{-n, 2 - n, 4 - n, \ldots, n\}$.

The PF of Y can then be found from the PF of X: $P(Y = k) = P(2X - n = k) = P(X = \frac{n+k}{2}) = \binom{n}{\frac{n+k}{2}}\left(\frac{1}{2}\right)^n$
if k is an integer between $-n$ and n (inclusive) such that $n + k$ is an even number. ■

If g is NOT one-to-one?

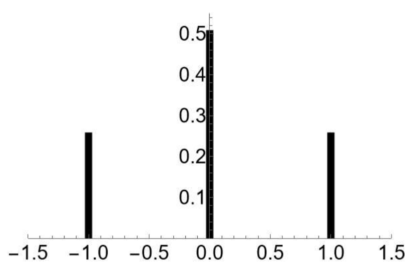
Then, if $g(x_1) = y$, there may be another x_2 such that $g(x_2) = y$ (!!)

$S \to X(s) \in \{x_1, x_2, x_3\}$. Then:



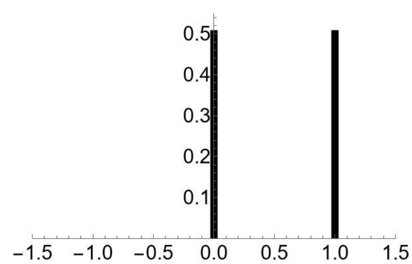
$\{x_1, x_2, x_3\} \to g(x_i)$

Thm (**PF of** $g(X)$). Let X be discrete and $g : \mathbb{R} \to \mathbb{R}$. Then the support of $g(X)$ is the set of all y such that $g(x) = y$ for at least one x in the support of X. The PF of $g(X)$ is:
$P(g(X) = y) = \sum_{x:g(x)=y} P(X = x)$, for all y in the support of $g(X)$.



$P(X = 0) = \frac{1}{2}$, $P(X = \pm 1) = \frac{1}{4}$

$\to$ $g(X) = X^2$ $\to$



$P(g(X) = 0) = \frac{1}{2}$, $P(g(X) = 1) = \frac{1}{2}$

Ex (**Not One to One**): Define X w/PF: $f_X(x) = \begin{cases} \frac{|x|}{32} & \text{for } x = -10, -5, 1, 0, 1, 5, 10 \\ 0 & \text{otherwise.} \end{cases}$

Let $Y = X^2$, what is the PF of Y?

Solution: Notice our Y values are $\{100, 25, 1, 0\}$ with 100, 25 and 1 getting double probabilities.

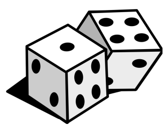
So $P(Y = 100) = 2 \cdot \frac{|10|}{32} = \frac{10}{16}$, $P(Y = 25) = 2 \cdot \frac{|5|}{32} = \frac{5}{16}$, and $P(Y = 1) = 2 \cdot \frac{|1|}{32} = \frac{1}{16}$

$$\text{So, } f_Y(y) = \begin{cases} \frac{\sqrt{y}}{16} & \text{for } y = 100, 25, 1, 0 \\ 0 & \text{otherwise.} \end{cases}$$

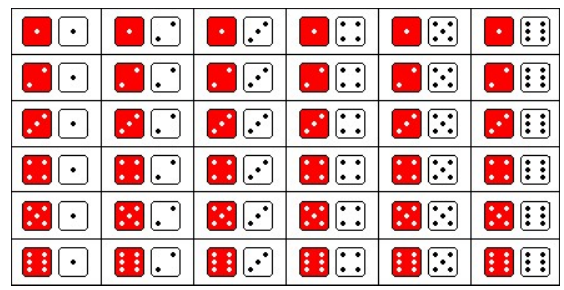
Sanity check: $\frac{\sqrt{100}}{16} + \frac{\sqrt{25}}{16} + \frac{\sqrt{1}}{16} + \frac{\sqrt{0}}{16} = 1.$ □

Def (**Function of Two Rvs**). Given an experiment with sample space S, if X and Y map $s \in S$ to $X(s)$ and $Y(s)$ respectively, then $g(X, Y)$ is the rv that maps s to $g(X(s), Y(s))$.

Ex (**Maximum of Two Die Rolls**).



s	X	Y	$\max(X, Y)$
(1, 2)	1	2	2
(1, 6)	1	6	6
(2, 5)	2	5	5
(3, 1)	3	1	3
(4, 3)	4	3	4
(5, 4)	5	4	5
(6, 6)	6	6	6



$P(\max(X, Y) = 1) = \frac{1}{36}$.

$P(\max(X, Y) = 2) = \sum_{s:\max(s)=2} P(s)$

$= P(X = 2, Y = 2) + P(X = 1, Y = 2) + P(X = 2, Y = 1)$

$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{3}{36}$.

$P(\max(X, Y) = 3) = \frac{5}{36}$.

$P(\max(X, Y) = 4) = \frac{7}{36}$.

$P(\max(X, Y) = 5) = \frac{9}{36}$.

$P(\max(X, Y) = 6) = \frac{11}{36}$.

Note: $P(\max(X, Y) = 5) = P(X = 5, Y \leq 4) + P(X \leq 4, Y = 5) + P(X = 5, Y = 5)$

$= 2P(X = 5, Y \leq 4) + \frac{1}{36}$ (symmetry)

$= 2\left(\frac{4}{36}\right) + \frac{1}{36} = \frac{9}{36}$.

Common error: to confuse a rv w/its distr.

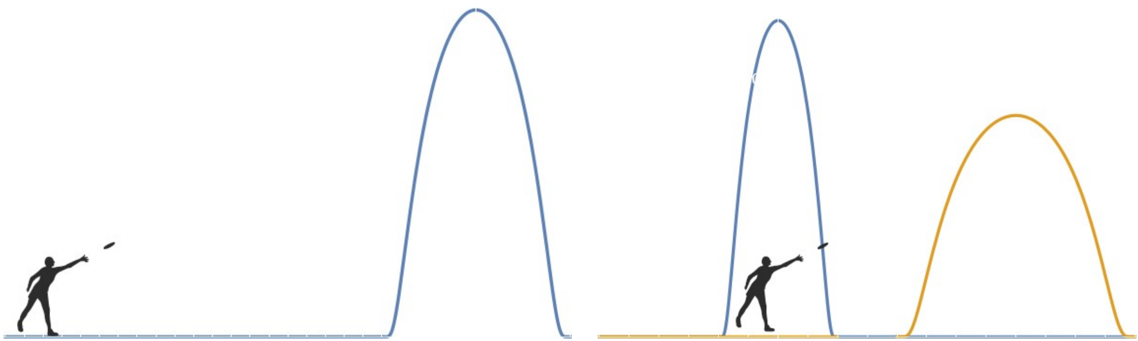
- The PF of $2X$ cannot be obtained by multiplying the PF of X by 2.
- If X, Y have the same distr, it's not (necessarily) true that $X = Y$.

§3.8 - Independence of Rvs

Similar to indep of *events* examined earlier. Intuitively, indep *rvs* X and Y means: if you know the value of X, this gives you no info about the value of Y.

For example, seeing the result of flipping a (fair) coin ($X = 1$, heads or $X = 0$, tails) gives you no info about the next flip (Y).

Def (**Indep of Two Cont. Rvs**): Continuous rvs X and Y are indep if $P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$, for **ALL** $x, y \in \mathbb{R}$.
(Recall $P(A,B) = P(A)P(B)$ for indep events)



Repeating from same spot, indep. Throw from previous landing spot, not indep.

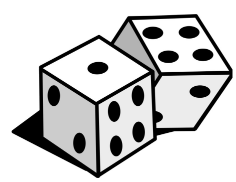
Def (**Indep of Two Discrete Rvs**): Discrete rvs X and Y are indep if $P(X = x, Y = y) = P(X = x)P(Y = y)$, for all x, y with x in the support of X and y in the support of Y.

Def (**Indep of Many Rvs**): Continuous rvs $X_1, \ldots, X_n$ are indep if $P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(X_1 \leq x_1) \ldots P(X_n \leq x_n)$, for *ALL* $x_1, \ldots, x_n \in \mathbb{R}$. (hence only one equation, as opposed to many eqs for events/discrete rvs)

For infinitely many cont. rvs, we say that they are indep if every finite subset of the rvs is indep.

☣ If $X_1, \ldots, X_n$ are indep, then they are pairwise indep, i.e., X_i is indep of X_j for $i \neq j$.
The idea behind proving that X_i and X_j are indep is to let all the x_k (other than x_i, x_j) go to ∞ in the definition of indep, since we already know $X_k < \infty$ is true (though it takes some work to give a complete justification for the limit).
But pairwise indep does not imply indep in general, as we saw in Chapter 2 for events.

Ex (**Dice Roll Indep**). In a roll of two fair dice, if X is the # on the first die and Y is the # on the second die, is $X + Y$ independent of $X - Y$??



Solution: No. Note that:

$$0 = P(X + Y = 12, X - Y = 1)$$

$$\neq P(X + Y = 12)P(X - Y = 1) = \tfrac{1}{36}\tfrac{5}{36}.$$

Def (iid): We'll often work with rvs that are indep and have the same distr.
We call such rvs indep and identically distributed, or iid for short.

Repeated throws are iid

Thm (Rv Function Indep): If X and Y are indep rvs, then any function of X is indep of any function of Y. For example X^2 would be indep from $\sqrt{\ln Y}$.

Thm (Binomial via Bernoulli): If $X \sim Bin(n,p)$, viewed as # of successes in n indep Bernoulli trials w/success prob p, we can write $X = X_1 + \ldots + X_n$ where the X_i are iid $Bern(p)$.

Proof. Let $X_i = 1$ if the ith trial was a success, and 0 if the ith trial was a failure.

It's like we have a person assigned to each trial, and ask each to raise their hand if their trial was a success.

If we count the raised hands (which is the same as adding up the X_i), we get the total # of successes. ■

Activity 7

Thm (Indep Binomial Addition). If $X \sim Bin(n,p)$, $Y \sim Bin(m,p)$, and X is indep of Y, then $X + Y \sim Bin(n + m,p)$.

Proofs (1 more in book)

Representation: Represent both X&Y as sum of i.i.d. $Bern(p)$ rvs:
$X = X_1 + \ldots + X_n$ and $Y = Y_1 + \ldots + Y_m$, where X_i and Y_j are all iid $Bern(p)$.

Then $X + Y$ is the sum of $n + m$ iid $Bern(p)$ rvs, so its distr, by previous thm, is $Bin(n + m,p)$. ■

Story: By the Binomial story, X is # of successes in n indep trials and Y is # of successes in m additional indep trials, all w/same success probability.

So $X + Y$ is total # of successes in the $n + m$ trials, which is the story of the $Bin(n + m,p)$ distr. ■

Def (Conditional Indep of Rvs). Rvs X and Y are conditionally indep given Z if for all $x,y \in \mathbb{R}$ and all z in the support of Z, $P(X \le x, Y \le y | Z = z) = P(X \le x | Z = z)P(Y \le y | Z = z)$.

For discrete rvs, an equivalent definition is to require: $P(X = x, Y = y | Z = z) = P(X = x | Z = z)P(Y = y | Z = z)$.

Def (Conditional PF). For any discrete rvs X and Z, the function $P(X = x | Z = z)$, when considered as a function of x for fixed z, is the conditional PF of X given $Z = z$.

❗ As with events, indep of rvs does not imply conditional indep (or vice versa).

Ex (**Matching pennies**). Consider a game called matching pennies. Each of two players, A and B, has a fair penny. They flip their pennies independently. If the pennies match, A wins; otherwise, B wins. Let X be 1 if A's penny lands Heads and -1 otherwise, and define Y similarly for B.

Let $Z = XY$, which is 1 if A wins and -1 if B wins. Then X and Y are unconditionally indep, but given $Z = 1$, we know $X = Y$ (the pennies match). So X and Y are conditionally dependent given Z. ☐

What did we learn?

- CDFs and Valid CDFs
- Functions of rvs
- Indep of rvs