Probability Theory

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Previous Lecture

- Sample Spaces
- Naive definition of prob: $\frac{\text{\# of outcomes favorable to } A}{\text{total \# of outcomes in } S}$
- How to count: multiplication rule
- Adjusting for over-counting

Counting Methods	**w/o repl**	**w/repl**
Order matters	$n(n-1)\dots(n-k+1)$	n^k
Order doesn't matter	$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$	$\binom{n+k-1}{k}$

§1.5 - Story Proofs: Proof by Interpretation

Story proofs can avoid messy calculations and go further than an algebraic proof toward explaining *why* the result is true.

❗ Story proofs are fully valid mathematically.

Ex (Team): For any nonnegative integers n and k with $k \leq n$, we have: $\binom{n}{k} = \binom{n}{n-k}$.

Story proof: Consider choosing a team of size k from a group of n people. We know there are $\binom{n}{k}$ possibilities.

Another way to choose is to specify the $n - k$ people who are *not* on the team.

Specifying who is on the team also determines who is not on the team, and vice versa.

So the two sides of $\binom{n}{k} = \binom{n}{n-k}$ are equal, as they are two ways of counting the same thing. ■

Ex (**Team Captain**): For any positive integers n&k with $k \leq n$, we have: $n\binom{n-1}{k-1} = k\binom{n}{k}$.

Story proof : Consider a group of n people, from which a team of k will be chosen, one of whom will be captain.

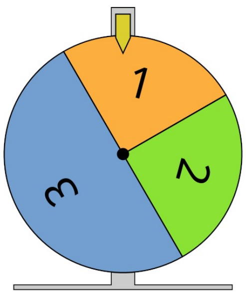
We could first choose the team captain and then choose the remaining $k-1$ team members; this and the multiplication rule gives us the left-hand side (LHS).

Equivalently, we could first choose k team members ($\binom{n}{k}$ ways), then choose one of them to be captain (k ways). This gives the right-hand side (RHS). ■

Harvard Video: youtube.com/watch?v=FJd_1H3rZGg&list=PL2SOU6wwxB0uwwH80KTQ6ht66KWxbzTIo&index=3

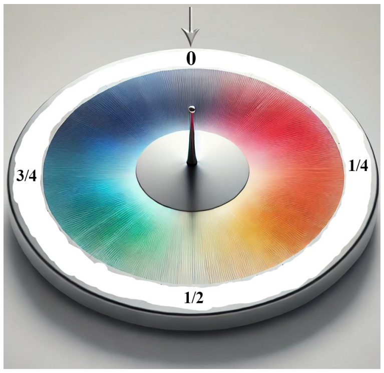
§1.6 - Non-Naive (General) Definition of Probability

What if the sample space S is infinite (e.g., $\mathbb{Z}$, $\mathbb{R}^2$, or $[0,1)$), or the outcomes are not equally likely? Ditch the naive definition!



Prob space w/spinning disk as random process.

Outcomes not equally likely.



$S = [0,1)$. Infinitely many outcomes.

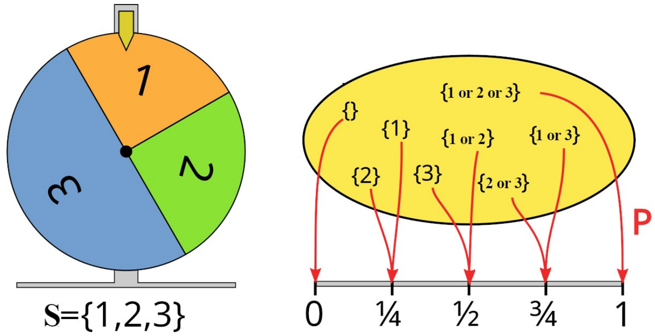
Def (**General Def of Probability**): A **prob space** consists of a **sample space** S and **prob function** P, which takes an event $A \subseteq S$ (as input) and returns $P(A)$, a real number between 0 and 1 (as output).

P must also satisfy axioms:

All-or-Nothing: $P(\emptyset) = 0, \quad P(S) = 1$.

Disjoint: If $A_1, A_2, \ldots$ are disjoint events, then: $P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$.

(Events being disjoint means they are mutually exclusive: $A_i \cap A_j = \emptyset$ for $i \neq j$.)

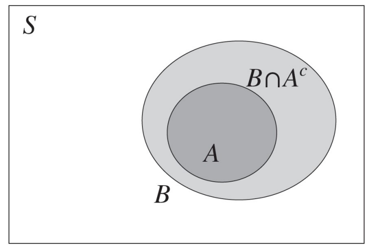


Thm (**Properties of Probability**): For any events A and B:

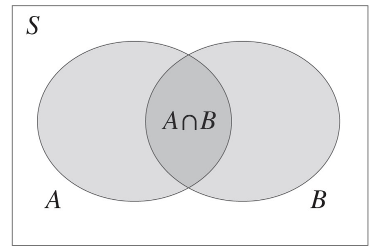
Complement: $P(A^c) = 1 - P(A)$.

Subset: If $A \subseteq B$, then $P(A) \leq P(B)$.

Inclusion-Exclusion: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.



Subset Probs

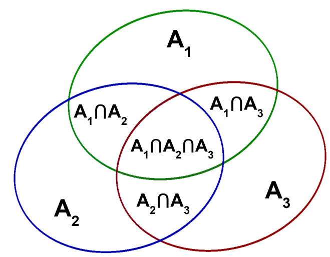


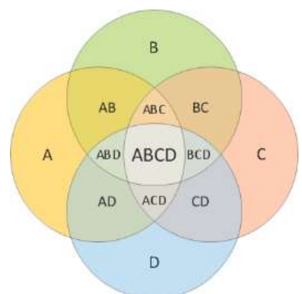
Inclusion/Exclusion Probs

[Proofs in book, they follow from axioms]

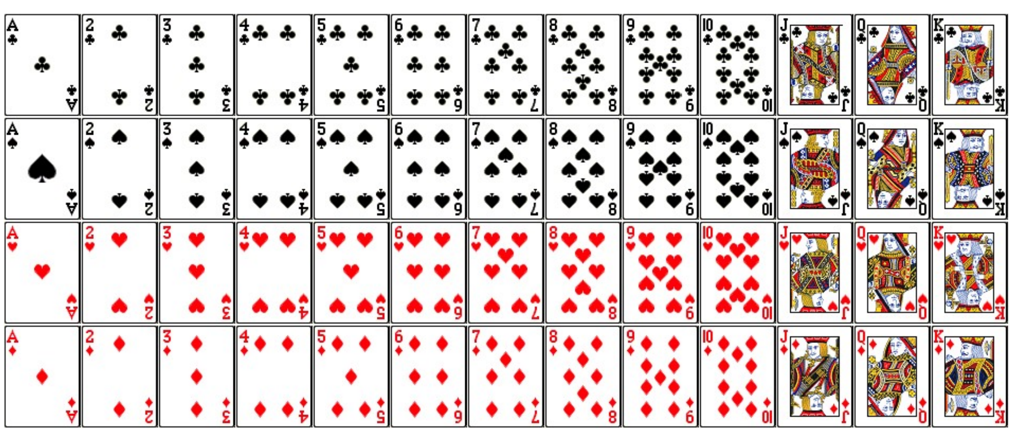
Thm (**Gen. Inclusion-Exclusion**): For any events $A_1, \ldots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1} P(A_1 \cap \ldots \cap A_n).$$





[Proof in Chpt 4]



Ex (**Void Suit**): A player is dealt a 13-card hand from a well-shuffled deck of cards. What's the prob the hand is void in at least one suit ("void in a suit" means having no cards of that suit)?

Solution: Let S, H, D, C be the events of being void in Spades, Hearts, Diamonds, Clubs, respectively.

We want to find $P(S \cup D \cup H \cup C)$. By inclusion-exclusion:

$$
\begin{aligned}
P(S \cup D \cup H \cup C) &= P(S) + \ldots + P(C) \\
&\quad - P(S \cap H) - \ldots - P(H \cap C) \\
&\quad + P(S \cap H \cap D) + \ldots + P(D \cap H \cap C) \\
&\quad - P(S \cap H \cap D \cap C)
\end{aligned}
$$

$$= 4P(S) - 6P(S \cap H) + 4P(S \cap H \cap D) - P(S \cap H \cap D \cap C). \qquad \text{(by symmetry)}$$

Prob of being void in a specific suit is:

$$P(S) = \frac{\text{\# of favorable outcomes}}{\text{total \# of outcomes}} = \frac{\text{choosing 13 cards from the 39 not in the suit}}{\text{choosing 13 cards from 52 in deck}} = \frac{\binom{39}{13}}{\binom{52}{13}}.$$

Prob of being void in 2 specific suits is:

$$P(S \cap H) = \frac{\binom{26}{13}}{\binom{52}{13}}.$$

Prob of being void in 3 specific suits is:

$$P(S \cap H \cap D) = \frac{\binom{13}{13}}{\binom{52}{13}} = \frac{1}{\binom{52}{13}}.$$

The last term $P(S \cap H \cap D \cap C)$ is 0 since it's impossible to be void in everything.

So prob is $4\frac{\binom{39}{13}}{\binom{52}{13}} - 6\frac{\binom{26}{13}}{\binom{52}{13}} + \frac{4}{\binom{52}{13}} \approx 0.051.$

Activity 4

Recall

Axioms of Probability

All-or-Nothing: $P(\emptyset) = 0, \quad P(S) = 1.$

Disjoint: If $A_1, A_2, \ldots$ are disjoint events, then:

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j).$$

Properties of Probability - For any events A and B:

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Subset: If $A \subseteq B,$ then $P(A) \leq P(B).$

Inclusion-Exclusion: $P(A \cup B) = P(A) + P(B) - P(A \cap B).$

What did we learn?

- Story Proofs
- General definition of probability
- General properties of probability