Theory of Probability Flashcards

These are flashcards made in preparation for oral exams involving the topics in probability: Random walks, Martingales, and Markov Chains. Textbook used: "Probability: Theory and Examples," Durrett.

Random Walks

Random Walk	Let $X_1, X_2, \dots$ be iid taking values in $\mathbb{R}^d$ and let $S_n = X_1 + \ldots + X_n$. S_n is a random walk.
Stopping Time	$(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ a filtered prob space. Stopping time $T : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is r.v. s.t. $\{T \leq n\} \in \mathcal{F}_n$ $\forall n \geq 0$, or equivalently, $\{T = n\} \in \mathcal{F}_n$ for all $n \geq 0$.
Stopping Time Examples	Constant times (e.g., $T \equiv 10$) are always stopping times. X_n an adapted process. Fix $A \in \mathcal{B}_{\mathbb{R}}$. Then first entry time into A, $T_A := \inf\{n \geq 0 : X_n \in A\}$, w/$\inf \emptyset := +\infty$ is stopping time
Stopping Times Closure Lemma	If S, T, T_n are stopping times on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$. Then so are: $S + T$, $S \wedge T := \min(S, T)$, $S \vee T := \max(S, T)$ $\liminf_n T_n$ and $\inf_n T_n$, $\limsup_n T_n$ and $\sup_n T_n$
Permutable Event	Given random seq. S and state space $\Omega := \{(\omega_1, \omega_2, \dots) : \omega_i \in S\}$ Event $A \in \mathcal{F}$ is permutable if $\pi^{-1}A \equiv \{\omega : \pi\omega \in A\} = A$, for any finite permutation π. $\varepsilon := \left\{A : A \text{ is permutable}\right\}$
Symmetric Function	$f : \mathbb{R}^n \to \mathbb{R}$ is symmetric if $f(x_1, x_2, \dots, x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$ for each $(x_1, \dots, x_n) \in \mathbb{R}^n$ and for each permutation $\pi \in \{1, 2, \dots, n\}$
Exchangeable σ-field	$X_1, X_2, \dots$ r.v.s on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $F_n := \left\{f : R^n \to R \text{ symmetric m'ble}\right\}$ Let $\varepsilon_n := \sigma(F_n, X_{n+1}, X_{n+2}, \dots)$. Exchangeable σ-field $\varepsilon := \bigcap_{n=1}^{\infty} \varepsilon_n$.
Hewitt Savage 0-1 **Law**	ε exchngble σ-field of iid $X_1, X_2, \dots$, $\mathcal{F} = \sigma(X_1, X_2 \dots)$, then $\mathbb{P}(A) \in \{0, 1\}$, $\forall A \in \varepsilon$
Random Walk Possibilities on $\mathbb{R}$	RWs on $\mathbb{R}$, 4 possibilities, one w/prob = 1. $S_n = 0\ \forall n$, $S_n \to \pm\infty$, or $-\infty = \liminf S_n < \limsup S_n = \infty$

RW Conv/Transients Thm	Convergence (divergence) of $\Sigma_n \mathbb{P}(\lvert S_n \rvert < \varepsilon)\ \forall \varepsilon > 0$ is sufficient to determine transience (recurrence) of S_n
RW Recurrence on $\mathbb{R}^d$	S_n recurrent in $d = 1$ if $S_n/n \xrightarrow{p} 0$. (or SSRW) S_n recurrent in $d = 2$ if $S_n/\sqrt{n}$ $\Rightarrow$non-deg. norm. dist. (or SSRW) S_n transient in $d \geq 3$ if is "truly three-dimensional"
Recurrence Thm for RWs	{recurrent values}=$\emptyset$ or is closed subgroup of $\mathbb{R}^d$. If closed subgroup, then {recurrent values}={possible values}
RW Equivalencies Thm (Hint: Recurrence)	Let $\tau_0 = 0$ and $\tau_n = \inf\{m > \tau_{n-1} : S_m = 0\}$ be time of nth return to 0 $\mathbb{P}(\tau_1 < \infty) = 1 \quad \Leftrightarrow \quad \mathbb{P}\big(S_m = 0 \text{ i.o.}\big) = 1 \quad \Leftrightarrow \quad \sum_{m=0}^{\infty} \mathbb{P}(S_m = 0) = \infty$
Wald's Identity	$\xi_1, \xi_2, \ldots$ be iid w/$\mu := \mathbb{E}[\xi_n] < \infty$. Set ξ_0 and let $S_n = \xi_1 + \ldots + \xi_n$ Let T be stopping time w/$\mathbb{E}[T] < \infty$. Then, $\mathbb{E}[S_T] = \mu \mathbb{E}[T]$
Recurrent Value	$x \in S$ is recurrent if, $\forall \varepsilon > 0$, we have $\mathbb{P}\big(\lvert S_n - x \rvert < \varepsilon \text{ i.o.}\big) = 1$
Possible Value (of RW)	S:={possible values}. $x \in S$ if for $\forall \varepsilon > 0$, $\exists n$ such that $\mathbb{P}(\lvert S_n - x \rvert < \varepsilon) > 0$.
Transient/Recurrent (RW)	If {recurrent values}=$\emptyset$, RW is transient, otherwise it is recurrent

Martingales

Conditional Expectation	$(\Omega, \mathcal{F}, P)$ w/$X \in L^1$, $G \subseteq \mathcal{F}$, Y:=$\mathbb{E}[X\|G]$ is unique s.t. Y is G-measurable and $\mathbb{E}\|Y\| < \infty$. $\mathbb{E}[\mathbb{E}[X\|G]1_A] = \mathbb{E}[Y1_A] = \mathbb{E}[X1_A]$, $A \in G$
$E[X\|A]$, **where** A **is an event is**:	Expected value of X given that A occurs
$E[X\|Y]$, **where** Y **is a r.v. is**:	r.v whose value at $\omega \in \Omega$ is $\mathbb{E}[X\|A]$ where A is the event $\{Y = Y(\omega)\}$
$\mathbb{E}[X\|1_A]$ **is**:	The case of $\mathbb{E}[X\|Y]$, for r.v. $Y = 1_A$, and $1_A(\omega)$ is 1 if $\omega \in A$ and 0 otherwise. It's a r.v that returns $\mathbb{E}[X\|A]$ if $\omega \in A$ and $\mathbb{E}[X\|A^c]$ if $\omega \notin A$
Absolute Continuity	Let ν and μ be σ-finite measures on $(\Omega, \mathcal{F})$. $\nu << \mu$, means that $\mu(A) = 0 \Rightarrow \nu(A) = 0$, for each $A \in \mathcal{F}$
Radon-Nikodym Lemma	Let ν and μ be two σ-finite measures on $(\Omega, \mathcal{F})$. $\nu << \mu \Leftrightarrow \exists$ $\mathcal{F}$-measurable $f : \Omega \to [0, \infty)$ s.t. $\nu(B) = \int_B f d\mu$, $\forall B \in \mathcal{F}$
If $X \in G$, **then** $E[X\|G] =$	X a.s.
If $G = \{\emptyset, \Omega\}$, **then** $E[X\|G] =$	$\mathbb{E}[X]$
If X **independent of** G, **then** $E[X\|G] =$	$\mathbb{E}[X]$ a.s.. To prove this, observe that $\mathbb{E}[X]$ is G-measurable and for any $A \in G$ we have: $\mathbb{E}[X1_A] = \mathbb{E}[X]\mathbb{E}[1_A] = \mathbb{E}[\mathbb{E}[X]1_A]$.
Pre-Tower Property If $\mathcal{F} \subset \mathcal{G}$ and $\mathbb{E}[X\|\mathcal{G}] \in \mathcal{F}$, then	$\mathbb{E}[X\|\mathcal{F}] = \mathbb{E}[X\|\mathcal{G}]$

Tower Property	Let $H \subseteq G$ be sub-σ-fields of $\mathcal{F}$. Then: $\mathbb{E}[\mathbb{E}[X\|G]\|H] = \mathbb{E}[X\|H]$ a.s.
Take out what is known If X is G-measurable, then for any r.v. Y s.t. $\mathbb{E}\|Y\| < \infty$ and $\mathbb{E}\|XY\| < \infty$, we have:	$\mathbb{E}[XY\|G] = X\mathbb{E}[Y\|G]$ a.s.
Conditional MCT	Let $X, X_n \geq 0$ be integrable r.v.s and $X_n \uparrow X$. Then $\mathbb{E}[X_n\|G] \uparrow \mathbb{E}[X\|G]$ a.s.
Conditional Jensen's Inequality	If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, $\mathbb{E}\|X\| < \infty$ and $\mathbb{E}\|\varphi(X)\| < \infty$, then $\mathbb{E}[\varphi(X)\|G] \geq \varphi(\mathbb{E}[X\|G])$ a.s.
L^p Contraction of Cond. Expectation	For $p \geq 1$, and $G \in \mathcal{F}$ $\mathbb{E}[\|\mathbb{E}[X\|G]\|^p] \leq \mathbb{E}[\|X\|^p]$. **Proof**: Jensen's $\Rightarrow$ $\|\mathbb{E}[X\|G]\|^p \leq E[\|X^p\| : G]$. Now take the expectation of both sides.
Conditional Fatou's Lemma	Let $X_n \geq 0$ be integrable r.v.s. and $\liminf_n X_n$ be integrable. Then $\mathbb{E}[\liminf_n X_n\|G] \leq \liminf_n \mathbb{E}[X_n\|G]$ a.s.
Conditional DCT	If $X_n \to X$ a.s. and $\|X_n\| \leq Y$ for some integrable r.v. Y. Then $\mathbb{E}[X_n\|G] \to \mathbb{E}[X\|G]$ a.s.
Chebyshev's Conditional Inequality	If $a > 0$, then $\mathbb{P}(\|X\| \geq a\|\mathcal{F}) \leq a^{-2}\mathbb{E}[X^2\|\mathcal{F}]$
Martingale (or sub, or super)	X_n on $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_n)$, s.t. X_n is adapted to $\mathcal{F}_n$. $\mathbb{E}\|X_n\| < \infty$ for each n. and, $\mathbb{E}[X_{n+1}\|\mathcal{F}_n] = X_n$ a.s.$\forall n$. (or $\geq$, or $\leq$ resp.)
If X_n is a martingale, **then for $n > m$, $\mathbb{E}[X_n\|\mathcal{F}_m] =$**	X_m
If X_n is martingale wrt $\mathcal{F}_n$ **and φ is convex, then**: (or sub)	If $\mathbb{E}\|\varphi(X_n)\| < \infty\ \forall n$, then $\varphi(X_n)$ is a sub-martingale wrt $\mathcal{F}_n$. Consequently, if $p \geq 1$ and $\mathbb{E}\|X_n\|^p < \infty\ \forall n$, then $\|X_n\|^p$ is a sub-martingale wrt $\mathcal{F}_n$.

Predictable Sequence	R.v.s H_n are predictible wrt $\mathcal{F}_n$ if it is $\mathcal{F}_{n-1}$ measurable for each $n \geq 1$.
Doob's Martingale Transform	Let $(X_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$–martingale, and H_n predictible. Transform is: $(H \cdot X)_0 = 0, \quad (H \cdot X)_n = \sum_{k=1}^{n} H_k(X_k - X_{k-1})$. If $(H \cdot X)_n$ integrable, then $(H \cdot X)_n$ is a martingale.
Doob's Mart Transform Lemma	Assume that X_n is a martingale and $(H \cdot X)_n \in L^1, \forall n$. Then, $H \cdot X$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.
Doob's Decomp	Submart X_n wrt $\mathcal{F}_n$ can be uniquely written as sum of mart M_n and increasing predictable process A_n. Let $D_0 = X_0, \ D_j = X_j - E[X_j \mid \mathcal{F}_{j-1}]$ $M_n = D_0 + D_1 + \ldots + D_n, \ A_0 = 0, \ A_n = X_n - M_n = E[X_n \mid \mathcal{F}_{n-1}] - (D_0 + \ldots + D_{n-1})$
Stopping Time SuperMartingale Prop	If T is a stopping time and $(X_n)_{n\geq 0}$ is a supermart then $(X_{T\wedge n})_{n\geq 0}$ is a supermart
Stopped Martingale Corollary	If T is a stopping time and $(X_n)_{n\geq 0}$ is a martingale then $(X_{T\wedge n})_{n\geq 0}$ is a martingale
Let T be a stopping time w/$E[T] < \infty$, then $E[T] =$	$\sum_{i=1}^{\infty} \mathbb{P}(T \geq i)$.
Doob's Upcrossing Inequality	Let $a < b$, and $U_n[a,b]$ the # of upcrossings from $a \to b$ by n. If X_n is submart, then $\mathbb{E}[U_n[a,b]] \leq \frac{\mathbb{E}[(X_n-a)^+]-\mathbb{E}[(X_0-a)^+]}{b-a}$
Martingale Convergence	Suppose that $(X_n)_{n\geq 0}$ is a sub-martingale with $\sup_n \mathbb{E}[X_n^+] < \infty$ Then for some X, we have $X_n \to X$ a.s., where $\mathbb{E}\lvert X\rvert < \infty$.
L^1-Bounded Martingale Convergence	If $(X_n)_{n\geq 0}$ is a martingale with $\sup_n \mathbb{E}\lvert X_n\rvert < \infty$, then $X_n \to X$ a.s. and $\mathbb{E}\lvert X\rvert < \infty$.
Non-negative Super-Mart Convergence	If $(X_n)_{n\geq 0}$ is a super-martingale with $X_n \geq 0$, then $X_n \to X$ a.s. and $\mathbb{E}[X] \leq \mathbb{E}[X_0]$

2nd Borel-Cantelli Lemma	Let $\mathcal{F}_n$ be filtration w/ $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and A_n events w/ $A_n \in \mathcal{F}_n$. Then, $\left\{A_n \text{ i.o.}\right\} = \left\{\sum_{n=1}^{\infty} \mathbb{P}(A_n \mid \mathcal{F}_{n-1}) = \infty\right\}$. If $A_n = X_n < \varepsilon \Rightarrow A_n \overset{a.s.}{\to} 0$. If $A_n = X_n > \varepsilon \Rightarrow X_n \overset{a.s.}{\nrightarrow} 0$.
Radon-Nikodym Martingale Thm	Let μ be finite, ν a prob. measure, $\mathcal{F}_n \uparrow \mathcal{F}$ be σ-fields, and μ_n, ν_n be restrictions of μ, ν to $\mathcal{F}_n$. If $\mu_n << \nu_n, \forall n$, and we let $X_n = d\mu_n/d\nu_n$. Then, X_n is a martingale wrt $\mathcal{F}_n$.
Galton-Watson Thm	ξ_i^n iid nonnegative integer r.v.s w/ $\mu := \mathbb{E}[\xi_i^n] \in (0,\infty)$. Let $Z_0 := 1$, $Z_{n+1} := \{\xi_1^n + \ldots + \xi_{Z_n}^n$, if $Z_n > 0$; or 0 otherwise. Then, $\frac{Z_n}{\mu^n}$ is a mart wrt $\mathcal{F}_n = \sigma(\xi_i^m : i \geq 1, 0 \leq m < n)$.
Galton-Watson Conclusions	If $\mu < 1$, then $Z_n = 0\ \forall n$ sufficiently large, so $Z_n/\mu^n \to 0$ If $\mu = 1$ and $\mathbb{P}(\xi_i^m = 1) < 1$, then $Z_n = 0$, $\forall n$ sufficiently large. If $\mu > 1$, then $\rho < 1$, that is, $\mathbb{P}\big(Z_n > 0 \text{ for all } n\big) > 0$.
Stopping Time Submart Ineq. (or mart)	If X_m is submart & T is stopping time w/ $\mathbb{P}(T \leq k) = 1$, for some $k \in \mathbb{Z}_+$, then $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] \leq \mathbb{E}[X_k]$. (or $\mathbb{E}[X_0] = \mathbb{E}[X_T] = \mathbb{E}[X_k]$ for mart)
Doob's Maximal Inequality	Let X_m be nonnegative submart, $X_n^* = \max_{0 \leq m \leq n} X_m$, $\lambda > 0$, and $A = \{X_n^* \geq \lambda\}$. Then, $\mathbb{P}(A) \leq \frac{1}{\lambda}\mathbb{E}[X_n 1_A] \leq \frac{1}{\lambda}\mathbb{E}[X_n]$.
$\mathbb{E}[X_n 1_A] = \mathbb{E}[X_{n-1} 1_A]$, $\forall A \in \mathcal{F}_{n-1} \Leftrightarrow$	$\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$.
L^p-Convergence Thm for Martingales	Suppose X_n is mart w/ $\sup \mathbb{E}[\lvert X_n \rvert^p] < \infty$ for some $p > 1$. Then, $X_n \to X$ a.s. and in L^p.
Uniform Integrability	Family of r.v.s $(X_\alpha)_{\alpha \in \Lambda}$ is uniformly integrable (*UI*) if $\sup_{\alpha \in \Lambda} \mathbb{E}[\lvert X_\alpha \rvert 1_{\{\lvert X_\alpha \rvert > M\}}] \to 0$ as $M \to \infty$. Remrk: Since $\mathbb{E}\lvert X_\alpha \rvert \leq M + \mathbb{E}[\lvert X_\alpha \rvert 1_{\{\lvert X_\alpha \rvert > M\}}]$, then $UI \Rightarrow L^1$-bounded
Sub σ-field UI Lemma	Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then, $\left\{\mathbb{E}[X \mid \mathcal{G}] : \mathcal{G} \text{ a } \sigma\text{-field } \subset \mathcal{F}\right\}$ is uniformly integrable.
Convergence in Prob Equivalency Thm	If $X_n \to X$ in probability, then *TFAE* : ♦ $\{X_n : n \geq 0\}$ is uniformly integrable ♦ $X_n \to X$ in L^1 ($\mathbb{E}\lvert X_n - X \rvert \to 0$) ♦ $\mathbb{E}\lvert X_n \rvert \to \mathbb{E}\lvert X \rvert < \infty$. [Note: L^1 convergence $\Rightarrow$ convergent in probability and *UI*]

Martingale Convergence in Probability Corollary	If $X_n \overset{p}{\to} X$, $(X_n)_{n\geq 0}$ is UI $\Leftrightarrow$ $X_n \overset{L^1}{\to} X$. $\lvert X_n \rvert \leq Y$ for some $Y \in L^1$, then $X_n \overset{L^1}{\to} X$
Sub-martingale Equivalencies Thm For a submart X_n, TFAE:	♦ $(X_n)_{n\geq 0}$ is UI. ♦ X_n converges a.s. and in L^1. ♦ X_n converges in L^1. Also, if $(X_n)_{n\geq 0}$ is a martingale, then ♦ $\exists$ integrable r.v. X so that $X_n = \mathbb{E}[X \lvert \mathcal{F}_n]$.
Levy's 0-1 Law	Suppose that $\mathcal{F}_n \uparrow \mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n)$. and $A \in \mathcal{F}_\infty$, then $\mathbb{E}[1_A \lvert \mathcal{F}_n] \to 1_A$ a.s.. From which you can conclude Kolmogorov's 0-1.
Levy's Forward Law	Suppose that $\mathcal{F}_n \uparrow \mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n)$. If $X \in L^1$, then $\mathbb{E}[X \lvert \mathcal{F}_n] \to \mathbb{E}[X \lvert \mathcal{F}_\infty]$ a.s. and in L^1.
Kolmogorov's 0-1 Law	$\xi_1, \xi_2, \ldots$ be independent r.v.s and $\mathcal{F}_n = \sigma(\xi_1, \xi_2, \ldots, \xi_n)$, $\forall n$. Let $\mathcal{T} = \cap_{k=1}^{\infty} \sigma(\xi_k, \xi_{k+1}, \ldots)$ be tail σ-field. Then $\forall A \in \mathcal{T}$, $\mathbb{P}(A) \in \{0,1\}$.
DCT for Filtered Conditional Expectation	Suppose $Y_n \to Y$ a.s. and $\lvert Y_n \rvert \leq Z$, $\forall n$ where $\mathbb{E}[Z] < \infty$. If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ then $\mathbb{E}[Y_n \lvert \mathcal{F}_n] \to \mathbb{E}[Y \lvert \mathcal{F}_\infty]$ a.s. $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F})$
Backward Martingale	Let $(\mathcal{F}_{-n})_{n\geq 0}$ be sub-σ-fields, w/ $\ldots \subseteq \mathcal{F}_{-2} \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0$. ♦ $X_{-n} \in \mathcal{F}_{-n}$ for each $n \in \mathbb{Z}_+$. ♦ $X_{-n} \in L^1$ for each $n \in \mathbb{Z}_+$. ♦ $\mathbb{E}[X_{-n} \lvert \mathcal{F}_{-(n+1)}] = X_{-(n+1)}$ for each $n \in \mathbb{Z}_+$.
Example of UI Martingale	For reverse martingale: clearly, $\mathbb{E}[X_0 \lvert \mathcal{F}_{-n}] = X_{-n}$ for each $n \in \mathbb{Z}_+$. Hence, if $(X_{-n})_{n\in\mathbb{Z}_+}$ is a reverse martingale, then it is UI. Proof: $\mathbb{E}[\lvert X_0 \rvert] < \infty$, so by Sub σ-field UI Lemma, $\mathbb{E}[X_0 \lvert \mathcal{F}_{-n}]$ is UI.
Convergence of Reverse Mart Thm	Let $(X_{-n})_{n\geq 0}$ be reverse mart. Then $X_{-n} \overset{n\to\infty}{\to} X_{-\infty}$ a.s. and in L^1. Moreover, $\mathbb{E}[X_0 \lvert \mathcal{F}_{-\infty}] = X_{-\infty}$ where $\mathcal{F}_{-\infty} = \cap_{n\in\mathbb{Z}_+} \mathcal{F}_{-n}$.
Levy's Backward Law	Let $Y \in L^1$. Suppose decreasing σ-fields $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \ldots$ and $\mathcal{G}_\infty = \cap_{n=0}^{\infty} \mathcal{G}_n$. Then, $\mathbb{E}[Y \lvert \mathcal{G}_n] \to \mathbb{E}[Y \lvert \mathcal{G}_\infty]$ a.s. and in L^1

Exchangeable Sequence	X_n, where for each n, $(X_1, X_2, \dots, X_n) \overset{d}{=} (X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)})$, $\forall$ permutations π.
de Finetti's Thm	If X_n are exchangeable, then, conditional on ε, we have $X_1, X_2, \dots$ are iid.
Optional Stopping σ-field $\mathcal{F}_T$	Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ and T be stopping time. Denote by $\mathcal{F}_T$, the σ-field of "events which occur prior to time T." In symbols: $\mathcal{F}_T := \left\{A \in \mathcal{F} : A \cap \{T \leq n\} \in \mathcal{F}_n,\ \forall n \geq 0\right\}$.
Optional Stopping Proposition	If T is stopping time, then $\mathcal{F}_T$ is σ-field & T is $\mathcal{F}_T$-measble If $S \leq T$ is stopping time, then $\mathcal{F}_S \subseteq \mathcal{F}_T$. Let T be stopping time w/$\mathbb{P}(T < \infty) = 1$ & X_n be adapted, then $X_T \in \mathcal{F}_T$
UI SubMart Stopping Time Closure	If $(X_n)_{n\geq 0}$ is *UI* sub-mart, then for any stopping time T, $(X_{T\wedge n})_{n\geq 0}$ is *UI*
UI SubMart Stopping Time Ineq.	If X_n is UI submart, then $\forall$ stopping time $T \leq \infty$, we have: $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] \leq \mathbb{E}[X_\infty]$, where $X_\infty = \lim X_n$
Optional Stopping Thm for SubMarts (or mart)	If S,T are stopping times w/$\mathbb{P}(S \leq T < \infty) = 1$, and $(X_{T\wedge n})_{n\geq 0}$ is UI submart, then $\mathbb{E}[X_T \mid \mathcal{F}_S] \geq X_S$ a.s. Consequently, $\mathbb{E}[X_S] \leq \mathbb{E}[X_T]$. (switch to ='s for mart)
Finite Differences Submartingale w/Stopping Times	Suppose X_n is a submart and $\mathbb{E}[\lvert X_{n+1} - X_n\rvert : \mathcal{F}_n] \leq B$ a.s If T is a stopping time w/$\mathbb{E}[T] < \infty$, then $X_{T\wedge n}$ is uniformly integrable and hence $\mathbb{E}[X_T] \geq \mathbb{E}[X_0]$
Nonneg SuperMart Stopping Time Thm	X_n is nonnegative supermart and $T \leq \infty$ is stopping time, then $\mathbb{E}[X_0] \geq \mathbb{E}[X_T]$ where $X_\infty = \lim X_n$
Asymmetric Simple RW w/generating fnct $\varphi(x) := \Sigma_{k\geq 0} p_k x^k$ w/$p_k := \mathbb{P}(\xi_i = k)$	$\xi_1, \xi_2, \dots$ iid, $S_n := \xi_1 + \dots + \xi_n$, $\mathbb{P}(\xi_i = 1) = p$, $\mathbb{P}(\xi_i = -1) = q \equiv 1 - p$, with $\frac{1}{2} > p < 1$ $\varphi(x) := \left(\frac{q}{p}\right)^x \Rightarrow \varphi(S_n)$ is mart. $T_x = \inf\{n : S_n = x\}$, $a < 0 < b \Rightarrow \mathbb{P}(T_a < T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}$ $a < 0 \Rightarrow \mathbb{P}(\min_n S_n \leq a) = \mathbb{P}(T_a < \infty) = \left(\frac{1-p}{p}\right)^{-a}$. $b > 0 \Rightarrow \mathbb{P}(T_b < \infty) = 1$ & $\mathbb{E}[T_b] = \frac{b}{2p-1}$

Mart Bounded Increments	Let $X_1, X_2, \ldots$ be a martingale with $\lvert X_{n+1} - X_n \rvert \leq M < \infty$. Let $C := \left\{ \lim X_n \text{ exists and finite} \right\}$, and $D := \left\{ \lim\sup X_n = +\infty \text{ and } \lim\inf X_n = -\infty \right\}$. Then, $P(C \cup D) = 1$

Markov Chains

Example such that $\sup_{n\geq 1}\mathbb{E}\lvert X_n\rvert<\infty$ but $(X_n)_{n\geq 1}$ are not uniformly integrable	Let $\Omega = [0,1]$ with Lebesgue measure, and $X_n = n \cdot 1_{[0,\frac{1}{n}]}$. Then the X_n are bounded in L^1, but not uniformly integrable.
Convergence in Probability	A sequence $\{X_n\}$ of random variables converges in probability towards the random variable X if for all $\varepsilon > 0$, we have: $\lim_{n\to\infty} P(\lvert X_n - X\rvert > \varepsilon) = 0$.
Convergence in Distribution (**Weak Convergence**):	Let X_n, X be r.v.s w/CDFs F_n & F resp. We say that $X_n \overset{d}{\to} X$ or $X_n \Rightarrow X$ if $F_n(x) \to F(x)\ \forall x$ where F continuous at x (C_F). If above holds, then $\pi_n \overset{d}{\to} \pi$, where π_n and π are distributions of X_n/X resp.
Convergence Almost Surely	To say that the sequence X_n converges a.s., almost everywhere, with probability 1, or strongly towards X means that $\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = 1$.
Markov Chain	An $(\mathcal{F}_n)_{n\geq 0}$-adapted stochastic process $(X_n)_{n\geq 0}$ taking values in $(S, \mathcal{S})$ is called a Markov chain if it has the **Markov Property**: $\mathbb{P}(X_{n+1} \in B\lvert\mathcal{F}_n) = \mathbb{P}(X_{n+1} \in B\lvert X_n)$ a.s. for each $B \in \mathcal{S}$, $n \geq 0$.
Markov Chain Transition Probability	We define a Markov chain's $(X_n)_{n\geq 0}$ transition probabilities $(p_n)_{n\geq 0}$ as $\mathbb{P}(X_{n+1} \in B\lvert\mathcal{F}_n) =: p_n(X_n, B)$ almost surely for each $n \geq 0$ and $B \in \mathcal{S}$.
Transition Matrix	probability of moving from i to j in one time step is $\mathbb{P}(j\lvert i) =: p_{ij}$, if we put these into a matrix, we have the transition matrix $p = [p_{ij}]$.
Time Homogeneous Markov Chain (finite dimensional, continuous state space)	A Markov chain in which the transition probabilities are all the same $p_n = p$ for all time $n \geq 0$.
Markov Chain Distributions	X_n is Markov w/trans. prob. $(p_n)_{n\geq 0}$ & init. dist. μ, then finite dimensional dist. are given by $\mathbb{P}(X_0 \in A_0, X_1 \in A_1, \ldots, X_k \in A_k)$ $= \int_{A_0} \mu(dx_0) \int_{A_1} p_0(x_0, dx_1) \ldots \int_{A_k} p_k(x_{k-1}, dx_k)$
Strengthened Markov Prop. **Let X_n be Markov w/init dist μ.** **X_n coordinate maps on $\left(S^{\mathbb{Z}_+}, \mathcal{S}^{\mathbb{Z}_+}, P_\mu\right)$**	$\mathcal{F}_n := \sigma\left(X_0, \ldots, X_n\right)$. $\theta : S^{\mathbb{Z}_+} \to S^{\mathbb{Z}_+}$ where $\theta(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$ For any bounded measurable function $f : S^{\mathbb{Z}_+} \to \mathbb{R}$, and any $k \geq 0$, $\mathbb{E}_\mu[f \circ \theta^k \lvert\mathcal{F}_k] = \mathbb{E}_{X_k}[f]$ $\mathbb{P}_\mu$- a.s.

Chapman-Kolmogorov Equation	$\mathbb{P}_x(X_{m+n} = z) = \Sigma_y \mathbb{P}_x(X_m = y)\mathbb{P}_y(X_n = z)$ for each $m, n \in \mathbb{Z}^+$.
Absorbing	A state a is called absorbing if $\mathbb{P}_a(X_1 = a) = 1$.
Strong Markov Property	For any bounded measurable function $f : S^{\mathbb{Z}_+} \to \mathbb{R}$ and for any stopping time T, $\mathbb{E}_\mu[f \circ \theta^T \vert \mathcal{F}_T] = \mathbb{E}_{X_T}[f]$ on $\{T < \infty\}$ $\mathbb{P}_\mu$- a.s.
Reflection Principle	Let $\xi_1, \xi_2, \ldots$ be iid w/distribution symmetric about 0. Let $S_n = \xi_1 + \ldots + \xi_n$. If $a > 0$, then $\mathbb{P}(\sup_{m \leq n} S_m > a) \leq 2\mathbb{P}(S_n > a)$.
***k*th Return to y**	Let $T_y^0 := 0$, and for $k \geq 1$, let $T_y^k := \inf\{n > T_y^{k-1} : X_n = y\}$, the time of the kth return to y.
ρ_{yz}	$\mathbb{P}_y(T_z < \infty)$
Finite kth Return Prob. to z starting at y :	For $k \geq 1$, $\mathbb{P}_y(T_z^k < \infty) = \rho_{yz}\rho_{zz}^{k-1}$.
Recurrent (for Markov)	A state $y \in S$ is called **recurrent** if $\rho_{yy} = 1$ and is called **transient** if $\rho_{yy} < 1$.
If y is recurrent, then $P_y\left(X_n = y \text{ i.o.}\right) =$	$\lim\limits_{k\to\infty} \mathbb{P}_y(T_y^k < \infty) = \lim_k \rho_{yy}^k = 1$.
If y is transient, then $P_y\left(X_n = y \text{ i.o.}\right)$	$= \lim_k \rho_{yy}^k = 0$.
Total number of visits to y by the Markov chain X_n is notated as $N(y) :=$	$\sum_{n=1}^{\infty} 1_{\{X_n = y\}}$.

A state x leads to, or is accessible from another state $y \neq x$, denoted by $x \to y$, if:	$\rho_{xy} > 0$ (or equivalently, for some $n \geq 1,\ p^n(x,y) > 0$). Formally, $x \to y$ if $\exists n_{xy} \geq 0$ such that $\mathbb{P}(X_{n_{xy}} = y \mid X_0 = x) = p_{xy}^{(n_{xy})} > 0$
Communicating Class	"$\leftrightarrow$" is an equivalence relation. Therefore, there is a partition C_1, C_2 of S, with each block C_i being referred to as a communicating class.
Irreducible Subset	A closed subset $A \subseteq S$ is called irreducible if $x \leftrightarrow y$ for all $x, y \in A$. By definition, each class is irreducible.
Irreducible Markov Chain	Markov *chain* is **irreducible** if it is possible to get to any state from any state. Formally, Markov chain is **irreducible** if its state space is a single communicating class, i.e., $x \leftrightarrow y,\ \forall x, y \in S$
Properties when x is recurrent and $\rho_{xy} > 0$	i) $\rho_{yx} = 1$, ii) y is recurrent, iii) $\rho_{xy} = 1$.
Closed Subset of States	We call a subset of states $A \subseteq S$ closed if $\rho_{xy} = 0$ for all $x \in A$ and $y \notin A$
Is a recurrent class C closed, open, neither?	Closed. :-)
In a finite state Markov chain, a class is recurrent (respectively transient) if and only if:	it is closed (respectively not closed).
Birth & Death Chains X_n on $\{0,1,2,\ldots\}$. $p_i := p(i,i+1),\ q_i := p(i,i-1)\ ,\ r_i := p(i,i)$ Let: $\varphi(0) := 0,\ \varphi(1) := 1$, and $\varphi(k+1) = ?$	For $X_n = k \geq 1,\ \varphi(k+1) = \varphi(k) + \frac{q_k}{p_k}(\varphi(k) - \varphi(k-1))$. For irreducible: $\varphi(m+1) = \varphi(m) + \prod_{j=1}^{m} \frac{q_j}{p_j}$ for $m \geq 1$, and $\varphi(n) = \sum_{m=0}^{n-1} \prod_{j=1}^{m} \frac{q_j}{p_j}$ for $n \geq 1$.
Birth Death Chain: **the state 0 is recurrent if and only if**	$\varphi(M) \to \infty$ as $M \to \infty$, that is: $\varphi(\infty) \equiv \sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{q_j}{p_j} = \infty$. If $\varphi(\infty) < \infty$, then $\mathbb{P}_x(T_0 = \infty) = \frac{\varphi(x)}{\varphi(\infty)}$.

Stationary/Invariant Measure μ	$\mu P = \mu : \mu(y) = \Sigma_{x\in S}\mu(x)p(x,y)$. ($\mu$ is left eigenvector of p). The last equation says $\mathbb{P}_\mu(X_1 = y) = \mu(y)$. Using the Markov property and induction, we have $\mathbb{P}_\mu(X_n = y) = \mu(y)\ \forall n \geq 1$.
Stationary/Invariant Distribution π	Stationary/invariant measure that is a probability measure. $\pi p = \pi : \pi(y) = \Sigma_{x\in S}\pi(x)p(x,y)$, and $\Sigma_{x\in S}\pi(x) = 1$. It represents a possible equilibrium for the chain.
Suppose p **is irreducible. A necessary and sufficient condition for the existence of a reversible measure is**	i) $p(x,y) > 0$ implies $p(y,x) > 0$, and ii) for any loop $x_0, \ldots, x_n = x_0$ with $\prod_{1\leq i\leq n} p(x_i, x_{i-1}) > 0,\ \prod_{i=1}^{n} \frac{p(x_{i-1},x_i)}{p(x_i,x_{i-1})} = 1$.
Recurrent Time in y $\mu_x(y) :=$	Define $\mu_x(y)$ as the expected time spent in y between visits to x.
Positive Recurrent	$\mathbb{E}_x[T_x] = \sum_{n=1}^{\infty} n\mathbb{P}(T_x = n) = \sum_{y\in S}\mu_x(y) < \infty$, and $\mathbb{P}_x(T_x < \infty) = 1$. **Positive Recurrent** $\Rightarrow$ Recurrent
Null-Recurrent	$x \in S$ is said to be null recurrent if $\mathbb{P}_x(T_x < \infty) = 1$, but $\mathbb{E}_x[T_x] = \infty$. If $\{X_n\}$ is **recurrent** but not **null recurrent** then it is called **positive recurrent**. X_n is null recurrent if all X_i are null recurrent.
If a chain is finite and irreducible, then there exists:	A unique stationary/invariant distribution π, and it is positive recurrent.
If $\{X_n\}$ **is positive recurrent, then for every** $x, y \in S$:	$\lim_{n\to\infty} p^n(x,y) = \pi(y) > 0$ where $\pi : S \to [0,1]$ is the stationary/invariant distribution. $p^n(x,y) := \frac{1}{n}\Sigma_n(X_n = y \mid X_0 = x)$
For an irreducible, positive recurrent Markov chain, what quality does the stat./invariant distribution π **have**?	It's unique!
For an irreducible and recurrent chain, the following are true.	♦ **Stat. measures** are unique up to constant multiples. ♦ μ a **stat. measure**$\Rightarrow$ $\mu(x) > 0,\ \forall x$. ♦ **Stat. dist.** π, if exists, is unique ♦ **Stat. measure** has infinite mass$\Rightarrow$**Stat. dist.** π cannot exist.
If π **is a stat/invariant distribution of a Markov chain and** $\pi(x) > 0$, **then**	then x is recurrent.

For an irreducible Markov chain, the following are equivalent.	i) There exists $x \in S$ that is positive recurrent. ii) There exists a stationary distribution π. iii) Every state is positive recurrent.
If p irreducible and has stat. dist. π, then any other stationary measure is	a multiple of π.
Doubly Stochastic	Prob. transition matrix $p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$ is doubly stochastic if $\Sigma_i p_{ij} = 1\ \forall j$ and $\Sigma_j p_{ij} = 1\ \forall i$. Uniform distribution is stat. dist. of $p \Leftrightarrow p_{ij}$ is doubly stochastic
Stationary Sequence	$(X_n)_{n\geq 0}$ is stationary if $(X_n, X_{n+1}, \ldots) \stackrel{d}{=} (X_0, X_1, \ldots),\ \forall n \geq 0$ or equivalently, $(X_n, X_{n+1}, \ldots, X_{n+m}) \stackrel{d}{=} (X_0, X_1, \ldots, X_m),\ \forall n, m \geq 0$ Exchangeable sequences are stationary.
Reversible Measure	measure μ such that $\mu(x)p(x,y) = \mu(y)p(y,x)$. Is always stationary since $\Sigma_{x\in S}\mu(x)p(x,y) = \Sigma_{x\in S}\mu(y)p(y,x) = \mu(y)$, i.e., it is invariant under multiplication by p.
Aperiodic Markov Chain	For x, $I_x := \{n \geq 1 : p_n(x,x) > 0\}$. Let d_x be the GCD of I_x x has period d_x. If every state of a Markov chain has period 1, then we call the chain aperiodic.
What could cause $d_x = d_y$?	If $x \leftrightarrow y$. In other words, if $\rho_{xy} > 0$ and $\rho_{yx} > 0$.
If $d_x = 1$, then $\exists n_0 \geq 1$ such that:	$p^n(x,x) > 0$ for all $n \geq n_0$. e.g., if $I_x = \{5, 7\}$.
An irreducible aperiodic Markov chain has the following property: for each $x, y \in S$, there exists:	$n_0 = n_0(x,y) \geq 1$ such that $p^n(x,y) > 0$ for all $n \geq n_0$.
Irreducible Aperiodic Markov X_n is Null Recurrent if:	$\{X_n\}$ is **recurrent** and $\lim_{n\to\infty} p_n(x,y) = 0$ for all $x, y \in S$.
Markov Chain Convergence Theorem	Consider irreducible, aperiodic Markov with stat. dist. π Then, $p^n(x,y) \to \pi(y)$ as $n \to \infty$, for all $x, y \in S$.

Total Variation Distance	For two probability measures μ, ν on S their total variation distance is given by: $d_{TV}(\mu,\nu) := 1/2 \sum_{x=S} \lvert\mu(x) - \nu(x)\rvert = sup_{A\subset S}\lvert\mu(A) - \nu(A)\rvert$
Coupled Markov Chain. **Let $\mu,\ \nu$ be prob. measures on countable S, & $(X_n, Y_n)_{n\geq 0}$ on product space $S \times S$.**	Chain is coupled if: i) marginals X_n & Y_n are Markov w/same p & init. dist. μ,ν resp. ii) $X_n = Y_n$ for $n \geq T$, where $T := \inf\{n \geq 0 : X_n = Y_n\}$.
Markov Recurrent Corollary A state X is Recurrent $\Leftrightarrow$	A state $x \in S$ is **recurrent** if and only if $\mathbb{E}_x[N(x)] = \sum_{n=1}^{\infty} p^n(x,x) = \infty$, where $N(y) := \sum_{n=1}^{\infty} 1_{\{X_n=y\}}$ is total # visits to y.
Asymptotic Density of Returns where $N_n(y) := \sum_{m=1}^{n} 1_{\{X_m=y\}}$, is # visits to y by n. Let $y \in S$ recurrent. Then $\lim\limits_{n\to\infty} \frac{N_n(y)}{n} =$	$\frac{1}{\mathbb{E}_y[T_y]} 1_{\{T_y<\infty\}}$ $\mathbb{P}_y$- a.s.
For a Markov chain and any $x, y \in S$, if $N(y) := \sum_{n=1}^{\infty} 1_{\{X_n=y\}}$ is total # visits to y, then we have $\mathbb{E}_x[N(y)] =$	$\frac{\rho_{xy}}{1-\rho_{yy}} = \sum_{n=1}^{\infty} p^n(x,y)$ (where we interpret $\frac{0}{0} = 0$, $\frac{c}{0} = +\infty$ for $c > 0$)
Markov Prob Calculations on Countable Space	X_n be Markov on countable set S w/transition matrix p & init. dist. μ a) $\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \mu(i_0)p_0(i_0,i_1)\ldots p_{n-1}(i_{n-1},i_n)$ b) $\mathbb{P}(X_n = j\lvert X_0 = i) = (p^n)(i,j)$. c) $\mathbb{P}(X_n = j) = \sum_{i\in S} \mu(i)(p^n)(i,j)$
To test whether a recurrent state is postive-recurent or null-recurrent, we compute the mean return time:	If $\mathbb{E}_x[T_x] = \sum_{n=1}^{\infty} np^n(x,x) = \infty$, is null-recurrent. And if $\mathbb{E}_x[T_x] < \infty$, is positive recurrent.
For a Markov chain and any $x, y \in S$, if $N(y) := \sum_{n=1}^{\infty} 1_{\{X_n=y\}}$ is total # visits to y, then we have $P_x(N(y) = k) =$	$\rho_{xy}\rho_{yy}^{k-1}(1 - \rho_{yy}^k)$
Consider Markov X_n started from stat. dist. π & trans. matrix p. Fix $N \geq 1$ & $Y_n := X_{N-n}$ for $n = 0, 1, \ldots, N$. Then:	$(Y_n)_{0\leq n\leq N}$ is a time-homogeneous Markov chain with initial distribution π and transition matrix q given by $q(x,y) = \frac{\pi(y)p(y,x)}{\pi(x)}$
Birth Death Chain: **For any $c \in R$, let $T_c = \inf\{n \geq 1 : X_n = c\}$, If $a < x < b$, then: $\mathbb{P}_x(T_a < T_b) =$**	$\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}$, and $\mathbb{P}_x(T_b < T_a) = \frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}$.

Stationary/Invariant Measure Theorem

Let x be a recurrent state. Then: $\mu_x(y) :=$

$\mathbb{E}_x\left[\sum_{n=0}^{T_x-1} 1_{\{X_n=y\}}\right] = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y, T_x > n)$,

is a stationary measure

Pairs of states x,y communicate,

denoted by $x \leftrightarrow y$, if:

$x \to y$ and $y \to x$.

In other words, if $\rho_{xy} > 0$ and $\rho_{yx} > 0$.

Suppose Markov irreducible & recurrent.

Let μ be stat. measure w/$\mu(y) > 0, \ \forall y \in S$.

If ν is another stat. measure, then

$\mu = c\nu$ for some $c > 0$.

Stat./Invariant Distribution π:

Suppose that S is finite and p is irreducible.

Then:

there exists a unique solution to $\pi p = \pi$

with $\Sigma_{i \in S}\pi(i) = 1$ and $\pi(i) > 0$ for all $i \in S$.

On a Markov chain, if C is a finite closed set,

then it contains...

at least one recurrent state.

In particular, a finite closed class C is recurrent.

Calculating Stat./Invariant Distribution

If p is irreducible and has stat. distribution π,

then $\pi(x) =$

$\frac{1}{\mathbb{E}_x[T_x]}$.

Birth Death Chain: If S irreducible, $\varphi \geq 0$

w/$E_x[\varphi(X_1)] \leq \varphi(x)$ for $x \notin F$ (finite set),

and $\lim_{x\to\infty} \varphi(x) \to \infty$ as $x \to \infty$, **then:**

The Markov chain X_n is recurrent.