11.1 Exercises - Solutions

Problem 1. Solve $y'' - xy = 0$ using a power series centered at $x = 0$.

Solution: Assume a power series solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Observe: $y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$.

Substituting into the DEQ: $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$

In order to compare these, I need to first move the x into the 2nd sum.

$x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$

Now, I need to get the same indexing.
This can be done a few ways, but I will aim for having all of the indexing result in x^n terms.

Having $n \to n+2$ in the first series gives us: $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} \to \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n$.

Having $n \to n-1$ in the 2nd series gives us: $\sum_{n=0}^{\infty} a_n x^{n+1} \to \sum_{n=1}^{\infty} a_{n-1}x^n$.

So our equation becomes: $\sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$.

Peeling off one term from the first series, so we have both series starting at the same index, gives us:

$0 = 2a_2 + \sum_{n=1}^{\infty} a_{n+2}(n+1)(n+2)x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 2a_2 + \sum_{n=1}^{\infty}[a_{n+2}(n+1)(n+2) - a_{n-1}]x^n$.

Comparing powers of x, we see for x^0 : $a_2 = 0$.

For x^n where $n \geq 1$: $a_{n+2}(n+1)(n+2) = a_{n-1}$ or $a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}$.

This is a 3-term recurrence relation, so we must set the first 2 terms to arbitrary constants: $a_0 = A$, $a_1 = B$.

So we have: $\left\{A, B, 0, \frac{A}{6}, \frac{B}{12}, 0, \frac{A}{180}, \frac{B}{504}, 0, \ldots\right\}$.

And $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots = A + Bx + \frac{A}{6}x^3 + \frac{B}{12}x^4 + \frac{A}{180}x^6 + \frac{B}{540}x^7 + \ldots$

$= A\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \ldots\right) + B\left(x + \frac{x^4}{12} + \frac{x^8}{540} + \ldots\right)$

Problem 2. Solve $y'' + xy' = 0$ using a power series centered at $x = 0$.

Solution: Assume a power series solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Observe: $y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$. and $y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$.

Substituting into the DEQ: $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} = 0$

In order to compare these, I need to first move the x into the 2nd sum.

$x\sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} a_n n x^n$

Now, I need to get the same indexing.
This can be done a few ways, but I will aim for having all of the indexing result in x^n terms.

Having $n \to n+2$ in the first series gives us: $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} \to \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n$.

So our equation becomes: $\sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n + \sum_{n=1}^{\infty} a_n n x^n = 0$.

Peeling off one term from the first series, so we have both series starting at the same index, gives us:

$0 = 2a_2 + \sum_{n=1}^{\infty} a_{n+2}(n+1)(n+2)x^n + \sum_{n=1}^{\infty} a_n n x^n = 2a_2 + \sum_{n=1}^{\infty} [a_{n+2}(n+1)(n+2) + na_n]x^n$.

Comparing powers of x, we see for $x^0 : a_2 = 0$.

For x^n where $n \geq 1$: $a_{n+2}(n+1)(n+2) = -na_n$ or $a_{n+2} = \frac{-na_n}{(n+1)(n+2)}$.

Since this recurrence relation is broken up into even or odd, we must set two arbitrary constants:
$a_0 = A$ and $a_1 = B$.

So we have: $a_2 = 0$, $a_4 = \frac{-2a_2}{12} = 0$, $a_6 = 0, \ldots$ (all zero)

For odd: $a_3 = \frac{-1 \cdot a_1}{6} = -\frac{B}{6}$, $a_5 = \frac{-3}{20}\left(-\frac{B}{6}\right) = \frac{B}{40}$, $a_7 = \frac{-5}{6 \cdot 7} a_5 = \frac{-5}{42}\left(\frac{B}{40}\right) = -\frac{B}{336}, \ldots$

And $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots = A + Bx - \frac{B}{6}x^3 + \frac{B}{40}x^5 - \frac{B}{336}x^7 + \ldots$