9.2 Exercises - Solutions

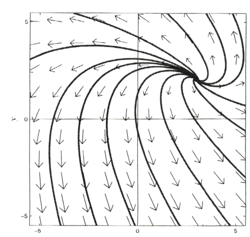
Problem 1 The system: $\frac{dx}{dt} = x - 2y + 1, \quad \frac{dy}{dt} = x + 3y - 9$, has a single critical point (x_0, y_0). Apply Theorem 2 (Stability of Almost Linear Systems, page 538) to classify this critical pt as to type and stability. Verify your conclusion by constructing a phase portrait for the given system.

Solution: Critical pts, $x - 2y + 1 = 0, \qquad x - 2y + 1 = 0$. Solving: $x = 2y - 1$ and $(2y - 1) + 3y - 9 = 5y - 10 = 0$ when $y = 2$. Therefore, $x = 2(2) - 1 = 3$.

Critical point: $(3, 2)$. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} \frac{\partial}{\partial x}(x - 2y + 1) & \frac{\partial}{\partial y}(x - 2y + 1) \\ \frac{\partial}{\partial x}(x + 3y - 9) & \frac{\partial}{\partial y}(x + 3y - 9) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.

$\mathbf{J}(3, 2)$ is still $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ and has characteristic equation $\lambda^2 - 4\lambda + 5 = 0$,

w/e-vals $\lambda_1, \lambda_2 = 2 \pm i$ that are complex conjugates with positive real part.



Hence the critical point $(3, 2)$ is an unstable spiral point as shown…

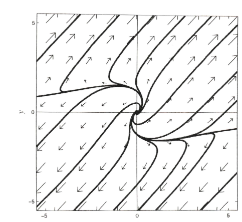
Problem 2 Investigate the type of the critical point $(0, 0)$ has for the almost linear system: $\frac{dx}{dt} = 5x - 3y + y(x^2 + y^2), \qquad \frac{dy}{dt} = 5x + y(x^2 + y^2)$.
Verify your conclusion by constructing a phase portrait.

Solution: We must first calculate the Jacobian matrix $\mathbf{J}$ and its eigenvalues at $(0, 0)$. Then we apply Theorem 2 to determine as much as we can about the type and stability of this critical points of the given almost linear system.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial}{\partial x}(5x - 3y + y(x^2 + y^2)) & \frac{\partial}{\partial y}(5x - 3y + y(x^2 + y^2)) \\ \frac{\partial}{\partial x}(5x + y(x^2 + y^2)) & \frac{\partial}{\partial y}(5x + y(x^2 + y^2)) \end{bmatrix} = \begin{bmatrix} 5 + 2xy & -3 + x^2 + 3y^2 \\ 5 + 2xy & x^2 + 3y^2 \end{bmatrix}$$

$\mathbf{J}(0, 0) = \begin{bmatrix} 5 & -3 \\ 5 & 0 \end{bmatrix}$. Has characteristic equation $\lambda^2 - 5\lambda + 15 = 0$,

and complex conjugate eigenvalues $\lambda_1 \approx 2.5 + 2.96i, \qquad \lambda_2 \approx 2.5 - 2.96i$.



Thus $(0, 0)$ is a spiral source of the almost linear system. The figure shows the critical pt:

Problem 3 The term bifurcation generally refers to something "splitting apart." With regard to DEQss or systems involving a parameter, it refers to abrupt changes in the character of the sols as the parameter is changed continuously. The problem below illustrates a sensitive case in which small perturbations (changes) in the coefficients of this almost linear system can change the type or stability (or both) of a critical pt.

$$\frac{dx}{dt} = -x + \varepsilon y, \qquad \frac{dy}{dt} = x - y.$$

Show that the critical pt $(0,0)$ is: $a)$ a stable spiral pt if $\varepsilon < 0$; $b)$ a stable node if $0 \le \varepsilon < 1$.

Solution: $\mathbf{J}(x,y) = \begin{bmatrix} -1 & \varepsilon \\ 1 & -1 \end{bmatrix}$. In this case, $\mathbf{J}(0,0) = \mathbf{J}(x,y) = \begin{bmatrix} -1 & \varepsilon \\ 1 & -1 \end{bmatrix}$.

$$\begin{vmatrix} -1-\lambda & \varepsilon \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 - \varepsilon = 0. \qquad \lambda = \frac{-2 \pm \sqrt{4-4(1-\varepsilon)}}{2} = -1 \pm \sqrt{\varepsilon}\,.$$

Recall: "Show that the critical pt $(0,0)$ is: $a)$ a stable spiral pt if $\varepsilon < 0$; $b)$ a stable node if $0 \le \varepsilon < 1$."

$a)$ If $\varepsilon < 0$ then $\lambda_1, \lambda_2 = -1 \pm i\sqrt{-\varepsilon}$. Thus the characteristic roots are complex conjugates w/negative real part, so it follows that $(0,0)$ is an asymptotically stable spiral pt.

$b)$ If $\varepsilon = 0$ then the characteristic roots $\lambda_1 = \lambda_2 = -1$ are equal and negative, so $(0,0)$ is an asymptotically stable node (star). Otherwise, If $0 < \varepsilon < 1$ then $\lambda_1, \lambda_2 = -1 \pm \sqrt{\varepsilon}$ are negative and unequal, so $(0,0)$ is an asymptotically stable improper node.

Problem 4 **What types of eigenvalues might be associated with these phase portraits of NONLINEAR systems**?

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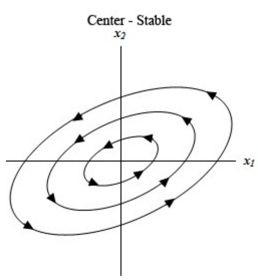
<td>
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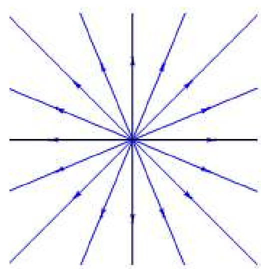
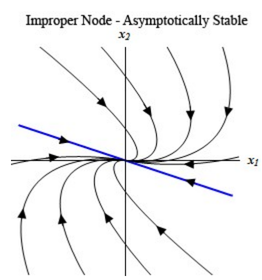
<td></td>
<td>
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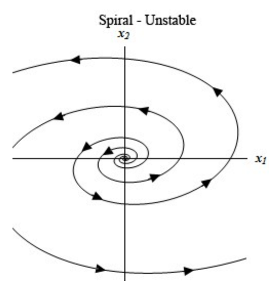
<td>
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<tr>
<td>Purely imaginary λ</td>
<td>Repeated λ, two e-vecs</td>
<td>Repeated negative λ w/one e-vec</td>
<td>Complex with positive real
OR purely imaginary</td>
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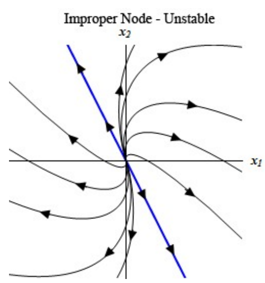
<td>
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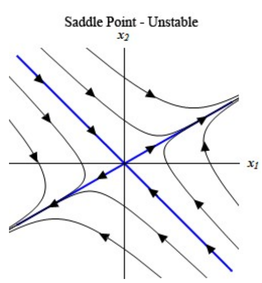
<td>
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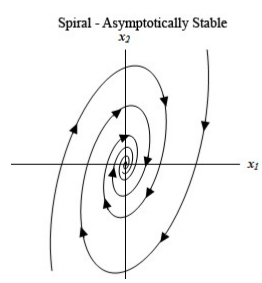
<td>
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<tr>
<td>Repeated positive λ w/one e-vec</td>
<td>Real with opposite signs
OR with one or both $\lambda = 0$</td>
<td>Complex with negative real
OR purely imaginary</td>
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[Adapted from Differential Equations & Linear Algebra, Edwards & Penny]