9.1 Exercises - Solutions

Problem 1 Find the critical point or points of $\frac{dx}{dt} = 1 - y^2, \qquad \frac{dy}{dt} = x + 2y$.

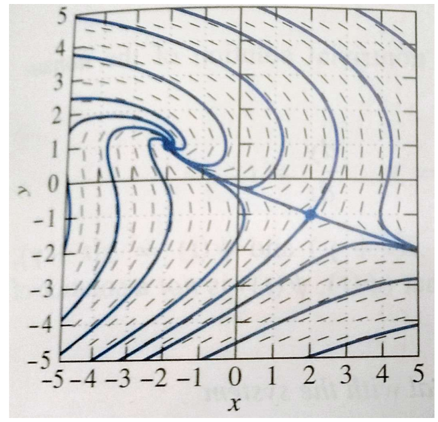
Solution: $0 = 1 - y^2 \qquad 0 = x + 2y$

Gives $y = 1$ or $y = -1$.

$0 = x + 2(1)$ or $0 = x + 2(-1)$ gives $x = -2$ or $x = 2$, respectively.

So, $(-2, 1)$ and $(2, -1)$ are our critical points.

See the graph…



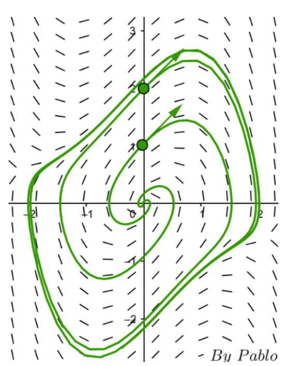
Problem 2 Find the equilibrium solution $x(t) \equiv x_0$ of: $x'' + (x^2 - 1)x' + x = 0$. Construct a phase portrait and direction field (Geogebra.org/m/utcMvuUy) for the equivalent 1st-order system: $x' = y, \quad y' = -(x^2 - 1)y - x$. Ascertain whether the critical point $(x_0, 0)$ looks like a center, saddle point, or spiral point.

Solution: Note that this equivalent system has $y := x'$.
And since an equilibrium does not change, we set $x' = x'' = y' = 0$, solve for x, y.
Substituting these into our system: $0 = y, \quad 0 = -(x^2 - 1) \cdot 0 - x \quad \Rightarrow \quad (x, y) = (0, 0)$.

Thus, for the original DEQ, we have the single equilibrium solution $x(t) \equiv 0$.

Phase plane portrait: $x' = y, \quad y' = -(x^2 - 1)y - x$ is shown below.

We observe that the critical point $(0, 0)$ in the phase plane looks like a spiral source,
with the solution curves emanating from this source spiraling outward toward a closed curve trajectory.



Problem 3 Solve the following system to determine whether the critical point $(0,0)$ is stable, asymptotically stable, or unstable. Construct a phase portrait and direction field for the given system (Geogebra.org/m/utcMvuUy). Ascertain the stability or instability of this critical point, and identify it visually as a node, a saddle point, a center, or a spiral point. $\frac{dx}{dt} = y,\quad \frac{dy}{dt} = -5x - 4y$

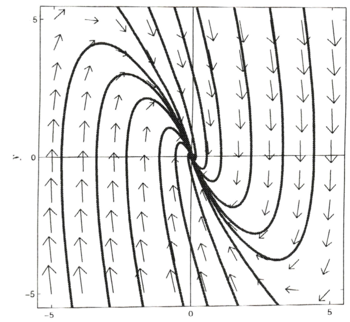
Solution: Substitution of $y' = x''$ from the first equation into the second one gives $x'' = -5x - y = -5x - 4x'$.

So $x'' + 4x' + 5x = 0$.

The characteristic roots of this equation are $r = -2 \pm i$, so we get the general solution…

$x(t) = e^{-2t}(A\cos t + B\sin t)$ and $y(t) =$?

$y(t) = e^{-2t}[(-2A + B)\cos t - (A + 2B)\sin t]$ (because $y = x'$).



Origin stable?

Clearly $x(t),\ y(t) \to 0$ as $t \to +\infty$ ($e^{-2t} \to 0$, while $\cos t$, $\sin t$ are bound between ± 1).
So the origin is stable.
Here you see the origin is an asymptotically stable spiral point w/ trajectories approaching $(0,0)$.

Problem 4 Given the system: $\frac{dx}{dt} = y^3 e^{x+y},\quad \frac{dy}{dt} = -x^3 e^{x+y}$.

Solve the equation: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-x^3 e^{x+y}}{y^3 e^{x+y}}$ to find the trajectories of the given system.

Construct a phase portrait and direction field for the system ((by hand, or Geogebra.org/m/utcMvuUy).

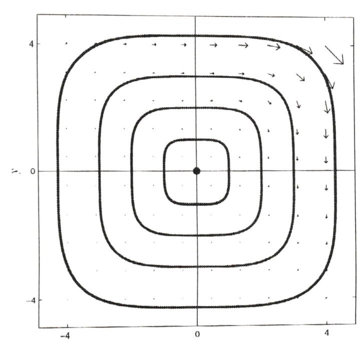
Identify visually the apparent character and stability of the critical point $(0,0)$ of the system.

Solution: $\frac{dy}{dx} = -\frac{x^3}{y^3}$

Separates to: $y^3 dy = x^3 dx$

$\int y^3 dy = \int x^3 dx \qquad \frac{1}{4}y^4 = \frac{1}{4}x^4 + C'$.

So $x^4 + y^4 = C$.



Closed periodic trajectories

Thus the trajectories consist of the origin $(0,0)$ and the ovals of the form $x^4 + y^4 = C$, as illustrated below…

So the critical pt is a stable center.