8.2 Exercises - Solutions 💪

Problem 1 Find a particular soln of $\vec{x}\,' = \begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}\vec{x} + \begin{bmatrix} e^t + t \\ 2e^t + 2 \end{bmatrix}$.

(assume the pre-trial you find is linearly independent from the homogenous sols)

Solution: We see the non-homogeneous part consists of polynomial and exponential forms:

$$\begin{bmatrix} e^t + t \\ -2e^t + 2 \end{bmatrix} = e^t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Therefore, we formulate the trial: $\vec{x}_{trial} = e^t\vec{c}_1 + t\vec{c}_2 + \vec{c}_3$,
where $\vec{c}_1 = (c_{11}, c_{12})$, $\vec{c}_2 = (c_{21}, c_{22})$ and $\vec{c}_3 = (c_{31}, c_{32})$.

Then, $\vec{x}\,'_{trial} = e^t\vec{c}_1 + \vec{c}_2$. And, $\mathbf{A}\vec{x}_{trial} = e^t\mathbf{A}\vec{c}_1 + t\mathbf{A}\vec{c}_2 + \mathbf{A}\vec{c}_3$, where $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$.

Putting it together: $e^t\vec{c}_1 + \vec{c}_2 = e^t\mathbf{A}\vec{c}_1 + t\mathbf{A}\vec{c}_2 + \mathbf{A}\vec{c}_3 + e^t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Writing this out so we can compare component-wise:

LHS: $e^t\vec{c}_1 + \vec{c}_2 = e^t\begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} + \begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} = \begin{bmatrix} c_{11}e^t + c_{21} \\ c_{12}e^t + c_{22} \end{bmatrix}$.

RHS: $e^t\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} + t\begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} c_{31} \\ c_{32} \end{bmatrix}$

$$= \begin{bmatrix} (4c_{11} + 2c_{12})e^t \\ (6c_{11} + 5c_{12})e^t \end{bmatrix} + \begin{bmatrix} (4c_{21} + 2c_{22})t \\ (6c_{21} + 5c_{22})t \end{bmatrix} + \begin{bmatrix} 4c_{31} + 2c_{32} \\ 6c_{31} + 5c_{32} \end{bmatrix}$$

$$= \begin{bmatrix} (4c_{11} + 2c_{12})e^t + (4c_{21} + 2c_{22})t + 4c_{31} + 2c_{32} \\ (6c_{11} + 5c_{12})e^t + (6c_{21} + 5c_{22})t + 6c_{31} + 5c_{32} \end{bmatrix}.$$

Adding in the rest of the RHS:

$$\begin{bmatrix} (4c_{11} + 2c_{12})e^t + (4c_{21} + 2c_{22})t + 4c_{31} + 2c_{32} \\ (6c_{11} + 5c_{12})e^t + (6c_{21} + 5c_{22})t + 6c_{31} + 5c_{32} \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} (4c_{11} + 2c_{12})e^t + (4c_{21} + 2c_{22})t + 4c_{31} + 2c_{32} + e^t + t \\ (6c_{11} + 5c_{12})e^t + (6c_{21} + 5c_{22})t + 6c_{31} + 5c_{32} + 2e^t + 2 \end{bmatrix}.$$

So we have: $c_{11}e^t + c_{21} = (4c_{11} + 2c_{12} + 1)e^t + (4c_{21} + 2c_{22} + 1)t + 4c_{31} + 2c_{32}$, and

$$c_{12}e^t + c_{22} = (6c_{11} + 5c_{12} + 2)e^t + (6c_{21} + 5c_{22})t + 6c_{31} + 5c_{32} + 2.$$

The exponential term gives us: $c_{11} = 4c_{11} + 2c_{12} + 1$ and $c_{12} = 6c_{11} + 5c_{12} + 2$.

The linear (t) terms give us: $0 = 4c_{21} + 2c_{22} + 1$ and $6c_{21} + 5c_{22} = 0$.

And the constant terms give us: $c_{21} = 4c_{31} + 2c_{32}$ and $c_{22} = 6c_{31} + 5c_{32} + 2$.

Solving the exponential equations gives: $\begin{bmatrix} -3 & -2 & | & 1 \\ -6 & -4 & | & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & -2 & | & 1 \end{bmatrix}$, $c_{12} = s$, $c_{11} = \frac{2s+1}{-3}$.

Thus: $(c_{11}, c_{12}) = (-1, 1)$ when $s = 1$.

Solving the linear terms: $\begin{bmatrix} 4 & 2 & | & -1 \\ 6 & 5 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & | & -1 \\ 2 & 3 & | & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -4 & | & -3 \\ 2 & 3 & | & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 3 & | & 1 \\ 0 & 4 & | & 3 \end{bmatrix}$, $c_{22} = \frac{3}{4}$, $2c_{21} + 3\left(\frac{3}{4}\right) = 1 \Rightarrow c_{21} = -\frac{5}{8}$.

Solving for the constant terms (using the previous calculation): $-\frac{5}{8} = 4c_{31} + 2c_{32}$, and $\frac{3}{4} = 6c_{31} + 5c_{32} + 2$.

$\begin{bmatrix} 4 & 2 & | & -\frac{5}{8} \\ 6 & 5 & | & -\frac{5}{4} \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & | & -\frac{5}{8} \\ 2 & 3 & | & -\frac{5}{8} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -4 & | & \frac{5}{8} \\ 2 & 3 & | & -\frac{5}{8} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & | & -\frac{5}{8} \\ 0 & 4 & | & -\frac{5}{8} \end{bmatrix}$

$c_{32} = -\frac{5}{32}$ and $2c_{31} = -3c_{32} - \frac{5}{8} = -3\left(-\frac{5}{32}\right) - \frac{5}{8} = -\frac{5}{32}$, thus $c_{31} = -\frac{5}{64}$.

So we have our particular solution: $\vec{x}_p = e^t\vec{c}_1 + t\vec{c}_2 + \vec{c}_3$,
where $(c_{11}, c_{12}) = (-1, 1)$, $(c_{21}, c_{22}) = \left(-\frac{5}{8}, \frac{3}{4}\right)$, and $(c_{31}, c_{32}) = \left(-\frac{5}{64}, -\frac{5}{32}\right)$.

Problem 2 Solve this DEQ using variation of parameters: $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\vec{x} + \begin{bmatrix} te^{2t} \\ 0 \end{bmatrix}$, where the soln of the associated homogeneous system gives the fundamental matrix: $\mathbf{X}(t) = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{bmatrix}$.

Solution: Recall from above, our general solution will be: $\vec{x}(t) = \mathbf{X}(t)\vec{c} + \mathbf{X}(t)\int_0^t \mathbf{X}(t)^{-1}\vec{f}(t)dt$

Focusing on the integral, note: $\mathbf{X}(t)^{-1} = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{bmatrix}^{-1} = e^{-3t}\begin{bmatrix} e^{2t} & -e^{2t} \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ 0 & e^{-2t} \end{bmatrix}$.

So, $\mathbf{X}(t)^{-1}\vec{f}(t) = \begin{bmatrix} e^{-t} & -e^{-t} \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} te^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} te^t \\ 0 \end{bmatrix}$.

Integrating, we get: $\int_0^t \begin{bmatrix} t \\ 0 \end{bmatrix} dt = \begin{bmatrix} \int_0^t te^t dt \\ 0 \end{bmatrix}$,

where $\int_0^t te^t dt = te^t|_0^t - \int_0^t e^t dt = te^t - [e^t]_0^t = te^t - (e^t - 1) = te^t - e^t + 1$. (integration by parts!)

And so: $\mathbf{X}(t)\int_0^t \mathbf{X}(t)^{-1}\vec{f}(t)dt = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} te^t - e^t + 1 \\ 0 \end{bmatrix} = \begin{bmatrix} te^{2t} - e^{2t} + e^t \\ 0 \end{bmatrix}$.