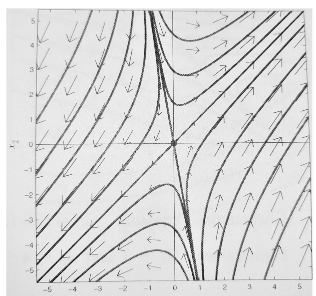
7.4 Exercises - Solutions 💪

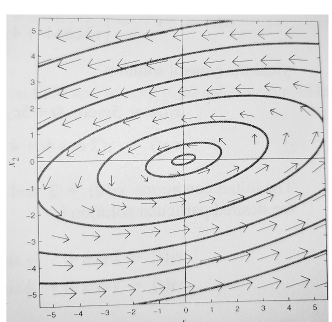
Problem 1 For the system: $x_1' = 4x_1 + x_2,\ x_2' = 6x_1 - x_2$, categorize the eigenvalues ($\lambda_{1,2} = -2, 5$) and eigenvectors ($\vec{v}_1 = [1 \quad -6]^T$ and $\vec{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$) of the coefficient matrix **A** according to Fig. 7.4.16 in the textbook, and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator (WolframAlpha.com: StreamPlot[{4x1 + x2, 6x1 - x2}, {x1, -2, 2}, {x2, -2, 2}]) to check your answer.

Solution: $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}$. With real eigenvalues on either side of zero, we expect a saddle point with trajectories moving outward along the e-vec $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ with the positive eigenvalue $\lambda = 5$, and trajectories moving inward along the eigenvector $\begin{bmatrix} 1 & -6 \end{bmatrix}^T$ with the negative eigenvalue $\lambda = -2$. And indeed, upon graphing we find:



Problem 2 For system: $x_1' = x_1 - 5x_2,\ x_2' = x_1 - x_2$, categorize e-vals ($\lambda_{1,2} = \pm 2i$) and e-vecs ($\vec{v}_{1,2} = [1 \pm 2i \quad 1]^T$) of **A** according to Fig. 7.4.16 in the textbook, and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator (WolframAlpha.com: StreamPlot[{x1 -5x2, x1 - x2}, {x1, -2, 2}, {x2, -2, 2}]) to check your answer.

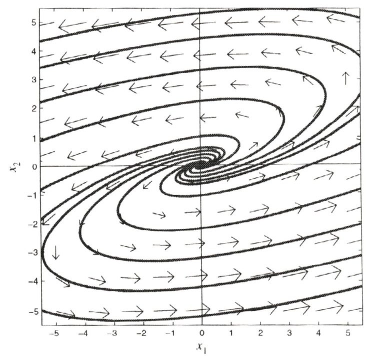
Solution: $\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$. With purely imaginary eigenvalues, we expect a center. And with $c = 1 > 0$, we expect the trajectories to be counterclockwise. And indeed, upon graphing we find:



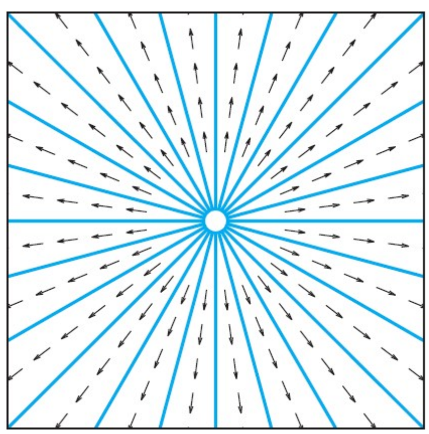
Problem 3 For the system: $x_1' = x_1 - 5x_2, \quad x_2' = x_1 + 3x_2$, categorize e-vals ($\lambda_{1,2} = 2 \pm 2i$) and e-vecs ($\vec{v}_{1,2} = [1 \pm 2i \quad -1]^T$) of **A** according to Fig. 7.4.16 in the textbook, and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator (WolframAlpha.com: StreamPlot[{x1 -5x2, x1 + 3x2}, {x1, -2, 2}, {x2, -2, 2}]) to check your answer.

Solution: $\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$. Observe that the real part of λ_i is greater than zero, so we expect a source.

Also, since the imaginary parts of λ_i are not zero, we expect a spiral. And with $c = 1 > 0$, we expect the spiral to be counterclockwise. And indeed, upon graphing we find:



Problem 4 The following phase portrait corresponds to a linear system of the form $\vec{x}' = \mathbf{A}\vec{x}$ in which the matrix **A** has two linearly independent eigenvectors. Determine the nature of the eigenvalues and eigenvectors. (For example, is it that the system has pure imaginary eigenvalues? Or does it has real eigenvalues of opposite sign; or an eigenvector associated with the positive eigenvalue is roughly $\begin{bmatrix} 2 & -1 \end{bmatrix}^T$, etc.)



Solution: Observe that there is no circular motion, so it appears that the eigenvalues are real. It also appears we have a source (arrows pointing out from the origin), and there are no trajectories that head towards the origin, so these real eigenvalues appear to be positive. Furthermore, each trajectory near the origin is tangent to a straight line, notably itself (since all of the trajectories are straight lines). Therefore, not only is it a node, it is a star point, or proper node.