7.2 Exercises - Solutions 💪

Problem 1 Given the system of DEQs below, given in matrix form: $x' = \mathbf{A}\vec{x}$. Verify that the given vector functions are solutions to that system. Then, use the Wronskian to show that the solutions are linearly independent. Finally, write the general solution to the system. $\vec{x}' = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}\vec{x}$;

Possible Solutions: $\vec{y}_1 = \begin{bmatrix} 3e^{-2t} \\ -2e^{-2t} \\ 2e^{-2t} \end{bmatrix}$, $\vec{y}_2 = \begin{bmatrix} e^{t} \\ -e^{t} \\ e^{t} \end{bmatrix}$, $\vec{y}_3 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ 0 \end{bmatrix}$.

Solution: Left Side: $\vec{y}_1' = -2e^{-2t}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = e^{-2t}\begin{bmatrix} -6 \\ 4 \\ -4 \end{bmatrix}$,

Right Side: $\begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}\vec{y}_1 = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}e^{-2t}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = e^{-2t}\begin{bmatrix} -6 \\ 4 \\ -4 \end{bmatrix}$.

Simarly: $\vec{y}_2' = e^{t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}e^{t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = e^{t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(I leave it to you to confirm the last vector is a solution).

"**Then, use the Wronskian to show that they are linearly independent**."

$$W(t) = \begin{vmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{vmatrix} = e^{3t}e^{t}e^{-2t}\begin{vmatrix} 3 & 1 & 1 \\ -2 & -1 & -1 \\ 2 & 1 & 0 \end{vmatrix} = e^{2t}\begin{vmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= e^{2t}(-1(0-1)) = e^{2t} \neq 0, \quad \text{for any } t.$$

"**Finally, write the general solution to the system**."

Since we have **three** linearly independent solutions for a system of **three** first order DEQs, we can write the general solution:

$$\vec{x}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 = c_1\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}e^{t} + c_3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}e^{3t}.$$

OR with different notation: $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 3c_1e^{-2t} + c_2e^t + c_3e^{3t} \\ -2c_1e^{-2t} - c_2e^t - c_3e^{3t} \\ 2c_1e^{-2t} + c_2e^t \end{bmatrix}$.

Problem 2 Find a particular solution to the system in the previous problem that satisfies the following initial conditions. $x_1(0) = 1,\ x_2(0) = 2,\ x_3(0) = 3$

Solution: $\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3c_1e^{-2\cdot 0} + c_2e^0 + c_3e^{3\cdot 0} \\ -2c_1e^{-2\cdot 0} - c_2e^0 - c_3e^{3\cdot 0} \\ 2c_1e^{-2\cdot 0} + c_2e^0 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 + c_3 \\ -2c_1 - c_2 - c_3 \\ 2c_1 + c_2 \end{bmatrix}$.

$$\mathbf{A}\vec{c} = (1,2,3) \Rightarrow \begin{bmatrix} 3 & 1 & 1 & 1 \\ -2 & -1 & -1 & 2 \\ 2 & 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -8 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & -1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -5 \end{bmatrix} \Rightarrow \vec{c} = (3,-3,-5).$$

So, $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 3c_1e^{-2t} + c_2e^t + c_3e^{3t} \\ -2c_1e^{-2t} - c_2e^t - c_3e^{3t} \\ 2c_1e^{-2t} + c_2e^t \end{bmatrix} = \begin{bmatrix} 3(3)e^{-2t} + (-3)e^t + (-5)e^{3t} \\ -2(3)e^{-2t} - (-3)e^t - (-5)e^{3t} \\ 2(3)e^{-2t} + (-3)e^t \end{bmatrix}$

$$= \begin{bmatrix} 9e^{-2t} - 3e^t - 5e^{3t} \\ -6e^{-2t} + 3e^t + 5e^{3t} \\ 6e^{-2t} - 3e^t \end{bmatrix}.$$

Problem 3 Suppose that one of the following vector functions is a constant multiple of the other on an open interval I: $\vec{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}$.

Show that their Wronskian $W(t) = |x_{ij}(t)|$ must vanish identically on I.

Solution: $\mathbf{W}(t) = \left|\vec{x}_1(t) \quad \vec{x}_2(t)\right|$. Also, $\vec{x}_2(t) = a_1\vec{x}_1(t) = \begin{bmatrix} a_1x_{11}(t) \\ a_1x_{21}(t) \end{bmatrix}$, for some constant a_1.

$$\mathbf{W}(t) = \left|\vec{x}_1(t) \quad \vec{x}_2(t)\right| = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} = \begin{vmatrix} x_{11}(t) & a_1x_{11}(t) \\ x_{21}(t) & a_1x_{21}(t) \end{vmatrix} \overset{C_2 - a_1C_1}{=} \begin{vmatrix} x_{11}(t) & 0 \\ x_{21}(t) & 0 \end{vmatrix} = 0.$$

And the constant function zero is equal to zero for all choices of t, so it "vanishes identically."