7.1 Exercises - Solutions 💪

Problem: #1 Transform the following system into an equivalent system of 1st-order DEQs.

$$x'' + 3x' + 4x - 2y = 0, \qquad y'' + 2y' - 3x + y = \cos t.$$

Solution: Dictionary, $x_0 := x,\ \ x_1 := x' = x_0',$ and $y_0 := y,\ \ y_1 := y' = y_0'.$

(note that you don't need the highest derivatives x'' or y'')

Therefore,

$x_0' = x_1, \qquad y_0' = y_1,$ (from the dictionary)

$x_1' = -4x_0 - 3x_1 + 2y_0, \qquad y_1' = 3x_0 - 2y_1 - y_0 + \cos t.$ (from the given DEQs)

I wrote the new DEQs this way anticipating the following form:

Let $\vec{v} := (x_0, x_1, y_0, y_1)$. Then $\vec{v}\,' = \mathbf{A}\vec{v} + \vec{f}$ OR

$$\begin{bmatrix} x_0' \\ x_1' \\ y_0' \\ y_1' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ y_0 \\ y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{bmatrix}.$$

Problem: #2 Find the general solution of the following system. Then, find the corresponding particular solution.

$$x' = -y, \qquad y' = 13x + 4y, \qquad x(0) = 0, \qquad y(0) = 3.$$

Solution: Note that they are coupled. Differentiating the first DEQ gives us:

$x'' = -y' = -(13x + 4y) = \ldots$

$\quad = -13x + 4x'.$ (we eliminated y !)

So, $x'' - 4x' + 13x = 0 \qquad r^2 - 4r + 13 = 0 \qquad r = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$

$e^{(2+3i)t} = e^{2t}(\cos 3t + i \sin 3t)$

$x(t) = e^{2t}(c_1 \cos 3t + c_2 \sin 3t)$

$x(0) = 0$ gives $0 = c_1 e^{2t}$, so $c_1 = 0$. Therefore, $x(t) = c_2 e^{2t} \sin 3t$.

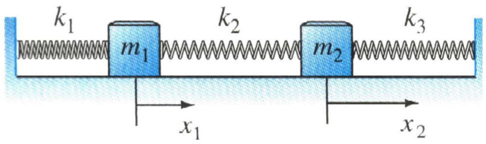
$y = -x'$ gives $y(t) = -(2c_2 e^{2t} \sin 3t + 3c_2 e^{2t} \cos 3t)$.

$y(0) = 3$, so: $3 = -3c_2$, and $c_2 = -1$.

Therefore, $x_p(t) = -e^{2t} \sin 3t$ and $y_p(t) = e^{2t}(2 \sin 3t + 3 \cos 3t)$

Problem: #3 Derive the DEQs: $m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$ and $m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2$

for the displacements (from equilibrium) of the two masses shown below (but don't try to solve them!):



Solution: Recall Newton's second law (for spring constant k): $mx'' = -kx$.

Looking at the above figure, we see that the first mass is pulled to the left (negatively) by the first spring and to the right (positively) by the second spring. The second mass is pulled to the left (negatively) by the second spring and to the right (positively) by the third spring. We also have to take into account the degree each spring is being stretched (to the right). For instance, the second spring is being stretched to the right a distance of x_2 by mass m_2, but the stretch is lessened by the distance x_1 has traveled to the right. So, the second spring is stretched: $x_2 - x_1$.

Hence for the first mass, Newton's second law gives: $m_1 x_1'' = -k_1(x_1) + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2 x_2$,

and similarly derived is the second mass: $m_2 x_2'' = -k_2(x_2 - x_1) + k_3(0 - x_2) = k_2 x_1 - (k_2 + k_3)x_2$.

Problem: #4 A particle of mass m moves in the plane with coordinates $\big(x(t),\ y(t)\big)$ under the influence of a force that is directed toward the origin and has magnitude $\frac{k}{x^2+y^2}$, an inverse-square central force field.

Show that: $mx'' = -\frac{kx}{r^3}$ and $my'' = \frac{ky}{r^3}$, where $r = \sqrt{x^2 + y^2}$.

Solution: Let θ be the polar angular coordinate of the point (x,y) and write $F = \frac{k}{x^2+y^2} = \frac{k}{r^2}$.

Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Therefore, $\cos\theta = \frac{x}{r}$, and $\sin\theta = \frac{y}{r}$ give the horizontal and vertical components of the vector lying on the unit circle pointing toward (x,y).

To have the correct direction, we need the force to point in the opposite direction (since gravity pulls toward origin), so we multiply by -1. And to have the correct magnitude we should multiply by $\frac{k}{r^2}$.

So, applying Newton's 2nd law gives:

$mx'' = -\frac{x}{r} \cdot \frac{k}{r^2} = -\frac{kx}{r^3}$, $\quad my'' = -\frac{y}{r} \cdot \frac{k}{r^2} = -\frac{ky}{r^3}$.