6.3 Exercises - Solutions

Problem 1 A city/suburban population transition matrix **A** is given.

Find the resulting long-term distribution of a constant total population between the city and its suburbs.

$$C_{k+1} = 0.8C_k + 0.1S_k \qquad \mathbf{A} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$
$$S_{k+1} = 0.2C_k + 0.9S_k$$

Solution: Characteristic polynomial, $p(\lambda) = \lambda^2 - \frac{17}{10}\lambda + \frac{7}{10} = \frac{1}{10}(\lambda - 1)(10\lambda - 7)$.

Eigenvalue: $\lambda_1 = 1,\ \lambda_2 = \frac{7}{10}$.

With $\lambda_1 = 1$: $\left.\begin{array}{c} -\frac{1}{5}x + \frac{1}{10}y = 0 \\ \frac{1}{5}x - \frac{1}{10}y = 0 \end{array}\right\} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

With $\lambda_2 = \frac{7}{10}$: $\left.\begin{array}{c} -\frac{1}{10}x + \frac{1}{10}y = 0 \\ \frac{1}{5}x + \frac{1}{5}y = 0 \end{array}\right\} \Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{10} \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

Recall **Transition Matrices**:
$\vec{x}_{k+1} = \mathbf{A}\vec{x}_k$ where $\vec{x}_0$ is the **initial vector**.
Therefore, $\vec{x}_k = \mathbf{A}^k\vec{x}_0$.

$$\vec{x}_k = \mathbf{A}^k \cdot \vec{x}_0 = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{10} \end{bmatrix}^k\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\vec{x}_0$$

$$\Rightarrow \quad \vec{x}_\infty = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\vec{x}_0$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} C_0 \\ S_0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} C_0 + S_0 \\ 2C_0 + 2S_0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} P_0 \\ 2P_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}P_0 \\ \frac{2}{3}P_0 \end{bmatrix},$$ where P_0 is the total initial population.

Thus the long-term distribution of population is $\frac{1}{3}$ city, $\frac{2}{3}$ suburban.

Problem 2 This problem deals with a fox-rabbit population.

Initially, there are F_0 foxes and R_0 rabbits. After k months, there are F_k foxes and R_k rabbits.

It involves the transition matrix $\mathbf{A} = \begin{bmatrix} 0.6 & 0.5 \\ -r & 1.2 \end{bmatrix}$ where the kill rate r is 0.175.

Show that in the long term, the populations of foxes and rabbits both die out.

Solution: Characteristic polynomial: $p(\lambda) = \lambda^2 - \frac{9}{5}\lambda + \frac{323}{400} = \frac{1}{400}(20\lambda - 19)(20\lambda - 17)$.

Eigenvalues: $\lambda_1 = \frac{19}{20}$, $\lambda_2 = \frac{17}{20}$.

With $\lambda_1 = \frac{19}{20}$: $\left.\begin{array}{l} -\frac{7}{20}x + \frac{1}{2}y = 0 \\ -\frac{7}{40}x + \frac{1}{4}y = 0 \end{array}\right\} \Rightarrow \vec{v}_1 = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$.

With $\lambda_2 = \frac{17}{20}$: $\left.\begin{array}{l} -\frac{1}{4}x + \frac{1}{2}y = 0 \\ -\frac{7}{40}x + \frac{7}{20}y = 0 \end{array}\right\} \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$\mathbf{P} = \begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \frac{19}{20} & 0 \\ 0 & \frac{17}{20} \end{bmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{4}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}$$

$$\vec{x}_k = \mathbf{A}^k \vec{x}_0 = \begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} \frac{19}{20} & 0 \\ 0 & \frac{17}{20} \end{bmatrix}^k \cdot \frac{1}{4}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}\vec{x}_0$$

$$\to \vec{x}_\infty = \frac{1}{4}\begin{bmatrix} 10 & 2 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 7 & -10 \end{bmatrix}\vec{x}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F_0 \\ R_0 \end{bmatrix}$$

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, as $k \to \infty$. Thus the fox and rabbit population both die out.

Problem 3 Suppose that $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Show that $\mathbf{A}^{4n} = \mathbf{I}$, $\mathbf{A}^{4n+2} = -\mathbf{I}$, and $\mathbf{A}^{4n+3} = -\mathbf{A}$, for every positive integer n.

Solution: $\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$.

So $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = -\mathbf{I}\mathbf{A} = -\mathbf{A}$,

$\mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = (-\mathbf{A})\mathbf{A} = -\mathbf{A}^2 = -(-\mathbf{I}) = \mathbf{I}$, and so forth.

Therefore, $\mathbf{A}^{4n} = \mathbf{I}$, $\mathbf{A}^{4n+2} = -\mathbf{I}$, and $\mathbf{A}^{4n+3} = -\mathbf{A}$, for every positive integer n.