

6.2 Exercises - Solutions

Problem 1 Determine whether $A = \begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix}$ is diagonalizable.

If it is, find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\text{Must use: } |\mathbf{A} - \lambda\mathbf{I}| = 0, \quad \begin{vmatrix} 6-\lambda & -5 & 2 \\ 4 & -3-\lambda & 2 \\ 2 & -2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -5 & 2 \\ 1-\lambda & -3-\lambda & 2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -5 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & -2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda) = 0. \quad \text{Eigenvalues: } \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

Recall: if an $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable!
(this is because $\lambda_1, \lambda_2, \lambda_3$ will necessarily produce three linearly independent eigenvectors.)

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad \text{Must use: } (\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}.$$

$$\text{With } \lambda_1 = 1 : \begin{bmatrix} 6-1 & -5 & 2 \\ 4 & -3-1 & 2 \\ 2 & -2 & 3-1 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 2 \\ 4 & -4 & 2 \\ 2 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow z = 0, y = b, x = b$$

$$\vec{v}_1 = \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ when } b = 1.$$

$$\text{With } \lambda_2 = 2 : \begin{bmatrix} 4 & -5 & 2 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix} \Rightarrow z = c, y = 0, x = -\frac{1}{2}c$$

$$\vec{v}_2 = \begin{bmatrix} -\frac{1}{2}c \\ 0 \\ c \end{bmatrix} = \frac{1}{2}c \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \text{ when } c = 2.$$

$$\text{With } \lambda_3 = 3 : \begin{bmatrix} 3 & -5 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 4 & -6 & 2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 6 & -6 \\ 0 & 4 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow z = c, y = c, x = c.$$

$$\vec{v}_3 = \begin{bmatrix} c \\ c \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ when } c = 1.$$

$$\text{So, } \mathbf{P} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}. \quad \text{And, } \mathbf{A} = \mathbf{PDP}^{-1}, \text{ where } \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Problem 2 Determine whether or not $\mathbf{A} = \begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ is diagonalizable.

If it is, find a diagonalizing matrix \mathbf{P} , and a diagonal matrix \mathbf{D} such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

$$\text{Must use: } |\mathbf{A} - \lambda\mathbf{I}| = 0. \quad \begin{vmatrix} -2-\lambda & 4 & -1 \\ -3 & 5-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 4 & -1 \\ 2-\lambda & 5-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 4 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda)^2. \quad \text{Eigenvalues: } \lambda_{1,2} = 1, \lambda_3 = 2. \quad (\text{diagonalizable?}) \quad \text{Must use: } (\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}.$$

$$\text{With } \lambda_{1,2} = 1 : \quad \left. \begin{array}{l} -3a + 4b - c = 0 \\ -3a + 4b - c = 0 \\ -a + b = 0 \end{array} \right\} \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad (\text{uh oh! only one.})$$

$$\text{With } \lambda_3 = 2 : \quad \left. \begin{array}{l} -4a + 4b - c = 0 \\ -3a + 3b - c = 0 \\ -a + b - c = 0 \end{array} \right\} \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(leave it to you to verify above as an exercise)

The matrix \mathbf{A} has only the two linearly independent e-vecs \vec{v}_1 and \vec{v}_2 , and therefore isn't diagonalizable.

Problem 3 Let \mathbf{A} be a 3×3 matrix with three distinct eigenvalues.

Tell how to construct six different invertible matrices $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_6$

and six different diagonal matrices $\mathbf{D}_1, \mathbf{D}_2, \dots, \mathbf{D}_6$ such that $\mathbf{P}_i\mathbf{D}_i(\mathbf{P}_i)^{-1} = \mathbf{A}$ for each $i = 1, 2, \dots, 6$.

Three eigenvectors associated with three distinct eigenvalues can be arranged with six different permutations as the column vectors of the invertible matrix: $\mathbf{P} = \begin{bmatrix} \vec{v}_i & \vec{v}_j & \vec{v}_k \end{bmatrix}$.

$$\mathbf{P}_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_3 & \vec{v}_2 \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} \vec{v}_2 & \vec{v}_1 & \vec{v}_3 \end{bmatrix},$$

$$\mathbf{P}_4 = \begin{bmatrix} \vec{v}_2 & \vec{v}_3 & \vec{v}_1 \end{bmatrix}, \quad \mathbf{P}_5 = \begin{bmatrix} \vec{v}_3 & \vec{v}_2 & \vec{v}_1 \end{bmatrix}, \quad \mathbf{P}_6 = \begin{bmatrix} \vec{v}_3 & \vec{v}_1 & \vec{v}_2 \end{bmatrix}.$$