6.1 Exercises - Solutions

Problem 1. Find the (real) eigenvalues, the associated eigenvectors, and a basis for each eigenspace for the matrix: $\mathbf{A} = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)((4-\lambda)(-1-\lambda)+6) \qquad \text{(pro tip....)}$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2) \qquad = \qquad -(\lambda-1)(\lambda-2)^2.$$

Characteristic Polynomial: $p(\lambda) = -(\lambda-1)(\lambda-2)^2 = 0$. **Eigenvalues**: $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$.

For each λ_k, solve $(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}$.

With $\lambda_1 = 1$:
$$\begin{bmatrix} 4-1 & -3 & 1 \\ 2 & -1-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1+(-1)R_2}{\Rightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_2+(-1)R_1}{\Rightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad z = 0,\ y = b,\ x = y = b.$$

$$\Rightarrow (b,b,1) = b(1,1,0). \qquad \vec{v}_1 = (1,1,0),\ \text{when } b = 1.$$

The eigenspace of $\lambda_1 = 1$ is 1-dimensional. Basis for λ_1 eigenspace: $\{\vec{v}_1\}$.

With $\lambda_{2,3} = 2$:
$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 4-2 & -3 & 1 \\ 2 & -1-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}$$

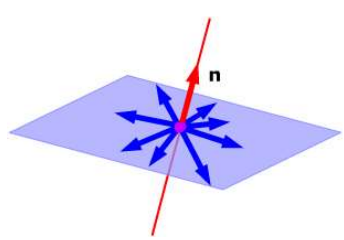
$$\Rightarrow \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, \quad z = c, \quad y = b, \quad x = \tfrac{3}{2}y - \tfrac{1}{2}z \ = \ \tfrac{3}{2}b - \tfrac{1}{2}c.$$

$$\Rightarrow \left(\tfrac{3}{2}b - \tfrac{1}{2}c, b, c\right) = b\left(\tfrac{3}{2},1,0\right) + c\left(-\tfrac{1}{2},0,1\right).$$

$\vec{v}_2 = (3,2,0)$ and $\vec{v}_3 = (-1,0,2)$, when $b,c = 2$

The eigenspace of $\lambda_{2,3} = 2$ is two-dimensional.

Basis for $\lambda_{2,3}$ eigenspace: $\{\vec{v}_2, \vec{v}_3\}$.



Problem 2. Find the complex-conjugate eigenvalues and corresponding eigenvectors of the matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix}.$$

Characteristic polynomial: $p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 0-\lambda & -12 \\ 12 & 0-\lambda \end{vmatrix} = \lambda^2 + 144 = 0.$

Eigenvalues: $\lambda_1 = -12i,\ \lambda_2 = +12i.$ For each λ_k, solve $(\mathbf{A} - \lambda_k\mathbf{I})\vec{v} = \vec{0}.$

With $\lambda_1 = -12i$: $\begin{bmatrix} 0-\lambda_1 & -12 \\ 12 & 0-\lambda_1 \end{bmatrix} = \begin{bmatrix} 12i & -12 \\ 12 & 12i \end{bmatrix} \overset{\frac{1}{12}R_{1,2}}{\Rightarrow} \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Rightarrow} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow\ y = b$ and $x = -ib$. So, $\vec{v}_1 = (-ib, b) = b(-i, 1) = (-i, 1),$ when $b = 1.$

Similarly with $\lambda_2 = +12i$: $\left.\begin{matrix} -12ia - 12b = 0 \\ 12a - 12ib = 0 \end{matrix}\right\}$ $\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$

Note that $\vec{v}_1$ and $\vec{v}_2$ are conjugate to each other.

Problem 3. **a) Suppose that A is a square matrix**.

Use the characteristic equation to show that A and $\mathbf{A}^T$ have the same eigenvalues.

Note first that $(\mathbf{A} - \lambda\mathbf{I})^T = (\mathbf{A}^T - \lambda\mathbf{I}^T) = (\mathbf{A}^T - \lambda\mathbf{I}),$ because $\mathbf{I}^T = \mathbf{I}.$

Since we learned earlier that the determinant of a square matrix equals the determinant of its transpose,

it follows that $|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{A}^T - \lambda\mathbf{I}|.$

This means the matrices **A** and $\mathbf{A}^T$ have the same characteristic polynomial. Therefore they have the same eigenvalues.

b) Give an example of a 2 × 2 matrix A such that A and $\mathbf{A}^T$ do not have the same eigenvectors.

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ with characteristic equation $(\lambda - 1)^2 = 0$ and the single eigenvalue $\lambda = 1.$

Then $\mathbf{A} - \mathbf{I} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and it follows that the only associated eigenvector is a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

The transpose $\mathbf{A}^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the same characteristic equation and eigenvalue,

but $\mathbf{A}^T - \mathbf{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so its only eigenvector is a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Thus **A** and $\mathbf{A}^T$ have the same eigenvalue but different eigenvectors.