

5.5 Exercises - Solutions

Problem 1 The roots of the equation $r^2 - 6r + 13 = 0$ are $r = 3 \pm 2i$.

Using the undetermined coefficients method, write down the **general form of a particular solution** for:

$$y'' - 6y' + 13y = xe^{3x} \sin 2x \text{ (this means you don't solve for the coefficients).}$$

$$e^{(3+2i)x} = e^{3x}(\cos 2x + i \sin 2x) \quad y_c = c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x.$$

$$y_i = (A + Bx)e^{3x} \sin 2x + (C + Dx)e^{3x} \cos 2x \quad \text{(pre-trial solution)}$$

Clearing up any linear dependence between y_i and y_c , we get: $y_{\text{trial}} = (Ax + Bx^2)e^{3x} \sin 2x + (Cx + Dx^2)e^{3x} \cos 2x$

General form of a Particular Solution:

$$y_g = y_c + y_{\text{trial}} = (c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x) + (Ax + Bx^2)e^{3x} \sin 2x + (Cx + Dx^2)e^{3x} \cos 2x.$$

Problem 2 Use the method of **variation of parameters** to find a particular solution to: $y'' - 4y = xe^x$.

Recall: y_p is of the form: $y_p = u_1 y_1 + u_2 y_2$.

So first need to obtain y_1, y_2 , from complementary solution. $r^2 - 4 = 0$, $r = \pm 2$, $y_c = c_1 e^{2x} + c_2 e^{-2x}$.

So, $y_1 = e^{2x}$, $y_2 = e^{-2x}$, and $y_1' = 2e^{2x}$, $y_2' = -2e^{-2x}$.

Variation of Parameters involves writing down: ($u_1' y_1 + u_2' y_2 = 0$) and ($u_1' y_1' + u_2' y_2' = f(x)$). So:

$$u_1' e^{2x} + u_2' e^{-2x} = 0, \quad (1) \quad 2u_1' e^{2x} - 2u_2' e^{-2x} = xe^x. \quad (2)$$

Solving for u_1' , u_2' , first start with the simpler equation (1): $u_1' = -\frac{u_2'}{e^{4x}}$. (3)

Plugging this into (2), and then solving for u_2' :

$$2\left(-\frac{u_2'}{e^{4x}}\right)e^{2x} - 2u_2' e^{-2x} = xe^x$$

$$-2\frac{u_2'}{e^{2x}} - 2\frac{u_2'}{e^{2x}} = -4\frac{u_2'}{e^{2x}} = xe^x$$

$$u_2' = -\frac{1}{4}xe^{3x}. \quad (4)$$

Plugging this into (3) to solve for u_1' : $u_1' = -\frac{u_2'}{e^{4x}} = -\frac{(-\frac{1}{4}xe^{3x})}{e^{4x}} = \frac{1}{4}xe^{-x}$. (5)

Now to integrate (4) and (5) to find u_1 and u_2 : $u_2 = \int u_2' dx = -\frac{1}{4} \int xe^{3x} dx$

$$= -\frac{1}{4} \left[\frac{1}{3}xe^{3x} - \frac{1}{3} \int e^{3x} dx \right] \quad \text{(using integration by parts: } \int u dv = uv - \int v du)$$

$$= -\frac{1}{4} \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} \right] = \left(\frac{1}{36} - \frac{1}{12}x \right) e^{3x}.$$

$u_1 = \int u_1' dx = \frac{1}{4} \int xe^{-x} dx = \frac{1}{4} \left[-xe^{-x} + \int e^{-x} dx \right] \quad \text{(using integration by parts: } \int u dv = uv - \int v du)$

$$= \frac{1}{4} \left[-xe^{-x} - e^{-x} \right] = -\left(\frac{1}{4}x + \frac{1}{4} \right) e^{-x}.$$

So our particular soln is: $y_p = u_1 y_1 + u_2 y_2 = \left[-\left(\frac{1}{4}x + \frac{1}{4} \right) e^{-x} \right] e^{2x} + \left[\left(\frac{1}{36} - \frac{1}{12}x \right) e^{3x} \right] e^{-2x}$

$$= -\left(\frac{1}{4}x + \frac{1}{4} \right) e^x + \left(\frac{1}{36} - \frac{1}{12}x \right) e^x = \left(\left(\frac{1}{36} - \frac{1}{12}x \right) - \left(\frac{1}{4}x + \frac{1}{4} \right) \right) e^x$$

$$= -\left(\frac{1}{3}x + \frac{2}{9} \right) e^x.$$

Even though the question didn't ask for it, our general soln would be: $y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - \left(\frac{1}{3}x + \frac{2}{9}\right)e^x$.

Alternatively, we can work this problem using the Wronskian method:

$$\text{Observe the Wronskian is: } W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4.$$

$$\begin{aligned} \text{So, } y_p &= -y_1 \int \frac{y_2 \cdot f(x)}{W} dx + y_2 \int \frac{y_1 \cdot f(x)}{W} dx = -e^{2x} \int \frac{e^{-2x} \cdot x e^x}{-4} dx + e^{-2x} \int \frac{e^{2x} \cdot x e^x}{-4} dx \\ &= \frac{1}{4} e^{2x} \int x e^{-x} dx - \frac{1}{4} e^{-2x} \int x e^{3x} dx \end{aligned}$$

$$\text{Recall: } \int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} = -(x+1)e^{-x}. \quad (\text{using integration by parts})$$

$$\text{and: } \int x e^{3x} dx = \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} = \left(\frac{1}{3}x - \frac{1}{9}\right) e^{3x}.$$

$$\text{So, } y_p = -\frac{1}{4} e^x (x+1) - \frac{1}{4} e^x \left(\frac{1}{3}x - \frac{1}{9}\right) = -\frac{1}{4} e^x \left(\frac{8}{9} + \frac{4}{3}x\right) = -\left(\frac{1}{3}x + \frac{2}{9}\right) e^x, \text{ as above.}$$

Problem 3 Find a particular solution y_p of $y^{(4)} - 5y'' + 4y = e^x - x e^{2x}$.

$$r^4 - 5r^2 + 4 = 0, \quad (r^2 - 4)(r^2 - 1) \quad r^2 = \{1, 4\}. \quad r = \pm 1, \pm 2.$$

$$y_c = c_1 e^{-x} + c_2 e^x + c_3 e^{-2x} + c_4 e^{2x}.$$

$$y_i = A e^x + (B + Cx) e^{2x} \quad (\text{pre-trial solution})$$

$$y_{\text{trial}} = A x e^x + B x e^{2x} + C x^2 e^{2x}$$

We need to determine the fourth derivative so that we can plug this into our original equation.

$$\begin{aligned} y'_{\text{trial}} &= (A x e^x + A e^x) + (B e^{2x} + 2B x e^{2x}) + (2C x e^{2x} + 2C x^2 e^{2x}), \\ &= (Ax + A)e^x + (2Bx + B)e^{2x} + (2Cx + 2C)x e^{2x}. \end{aligned}$$

$$y''_{\text{trial}} = (Ax + 2A)e^x + (4Bx + 4B)e^{2x} + (4Cx^2 + 8Cx + 2C)e^{2x}.$$

$$y'''_{\text{trial}} = (Ax + 3A)e^x + (8Bx + 12B)e^{2x} + (8Cx^2 + 24Cx + 12C)e^{2x}.$$

$$y^{(4)}_{\text{trial}} = (Ax + 4A)e^x + (16Bx + 32B)e^{2x} + (16Cx^2 + 64Cx + 48C)e^{2x}.$$

$$\text{Recall the original equation: } y^{(4)} - 5y'' + 4y = e^x - x e^{2x}.$$

$$\text{So the LHS trial version is: } (Ax + 4A)e^x + (16Bx + 32B)e^{2x} + (16Cx^2 + 64Cx + 48C)e^{2x}$$

$$- 5[(Ax + 2A)e^x + (4Bx + 4B)e^{2x} + (4Cx^2 + 8Cx + 2C)e^{2x}] + 4[Ax e^x + Bx e^{2x} + Cx^2 e^{2x}]$$

$$= (Ax + 4A)e^x + (16Bx + 32B)e^{2x} + (16Cx^2 + 64Cx + 48C)e^{2x}$$

$$- [(5Ax + 10A)e^x + (20Bx + 20B)e^{2x} + (20Cx^2 + 40Cx + 10C)e^{2x}] + [4Ax e^x + 4Bx e^{2x} + 4Cx^2 e^{2x}]$$

$$= -6A e^x + (12B + 38C)e^{2x} + 24C x e^{2x}.$$

$$- 6A e^x + (12B + 38C)e^{2x} + 24C x e^{2x} = e^x - x e^{2x}$$

$$- 6A = 1, \quad 12B + 38C = 0, \quad 24C = -1$$

$$A = -\frac{1}{6}, \quad C = -\frac{1}{24}, \quad 12B + 38\left(-\frac{1}{24}\right) = 0, \quad 12B = \frac{19}{12}, \quad B = \frac{19}{144}.$$

$$\text{Recall: } y_{\text{trial}} = A x e^x + B x e^{2x} + C x^2 e^{2x}. \text{ So our particular solution is: } y_p = -\frac{1}{6} x e^x + \frac{19}{144} x e^{2x} - \frac{1}{24} x^2 e^{2x}.$$

And even though the question did not ask for it, our general solution would be...

$$y = y_p + y_c = \left(-\frac{1}{6} x e^x + \frac{19}{144} x e^{2x} - \frac{1}{24} x^2 e^{2x}\right) + (c_1 e^{-x} + c_2 e^x + c_3 e^{-2x} + c_4 e^{2x}). \quad (\text{whew!})$$