5.3 Exercises - Solutions

Problem 1 Solve the initial value problem: $y^{(3)} + 10y^{(2)} + 25y' = 0$, with init-conds: $y(0) = 3$, $y'(0) = 4$, and $y''(0) = 5$.

$r^3 + 10r^2 + 25r = 0,$

$r(r^2 + 10r + 25) = 0,$

$r(r+5)^2 = 0.$

$r = 0$, and $r = -5$ with multiplicity 2.

So: $y = c_1 + c_2 e^{-5x} + c_3 x e^{-5x}.$ $\quad (*)$

Plugging in the initial condition $y(0) = 3$:

$3 = c_1 + c_2.$ $\quad (**)$

$c_1 = 3 - c_2 \quad \Rightarrow \quad y = 3 - c_2 + c_2 e^{-5x} + c_3 x e^{-5x}.$

$y' = -5c_2 e^{-5x} - 5c_3 x e^{-5x} + c_3 e^{-5x}.$

Plugging in the initial condition $y'(0) = 4$:

$4 = -5c_2 + c_3$ $\quad (***)$

$c_3 = 4 + 5c_2 \quad \Rightarrow \quad y' = -5c_2 e^{-5x} - (20 + 25c_2) x e^{-5x} + (4 + 5c_2) e^{-5x} = e^{-5x}(4 - 5(5c_2 + 4)x).$

$y'' = -5e^{-5x}(4 - 5(5c_2 + 4)x) + e^{-5x}(0 - 5(5c_2 + 4))$

Plugging in the initial condition $y''(0) = 5$:

$5 = -5(4 - 0) - 5(5c_2 + 4) \quad \Rightarrow \quad 5 = -40 - 25c_2 \quad \Rightarrow \quad c_2 = -\frac{45}{25} = -\frac{9}{5}.$

Plugging this into $(***)$, we get $c_3 = 4 + 5(-\frac{9}{5}) = -5.$

And plugging into $(**)$, we get $3 = c_1 - \frac{9}{5} \quad \Rightarrow \quad c_1 = \frac{24}{5}.$

And plugging all of these into our general soln $(*)$, we have the particular soln: $y_p = \frac{24}{5} + -\frac{9}{5} e^{-5x} - 5x e^{-5x}.$

Problem 2 **Find a general soln of** $y^{(3)} + 3y'' + 4y' - 8y = 0.$

$r^3 + 3r^2 + 4r - 8 = 0.$ Educated guesses: $\{\pm 1, \pm 2, \pm 4, \pm 8\}.$

$r = 1$ works.

So: $r^3 + 3r^2 + 4r - 8 = (r-1)r^2 + (4r^2 + 4r - 8)$

Further dividing the remainder:

$4r^2 + 4r - 8 = (r-1)4r + (8r - 8) = (r-1)4r + 8(r-1) = (r-1)(4r+8).$

Therefore, $r^3 + 3r^2 + 4r - 8 = (r-1)r^2 + (r-1)(4r+8) = (r-1)(r^2 + 4r + 8).$

$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 8}}{2} = -2 \pm 2i.$ So, $r \in \{1, -2 \pm 2i\}.$

$e^{(-2+4i)x} = e^{-2x}(\cos 4x + i \sin 4x).$

Linearly independent sols from above (real and imaginary parts) are: $e^{-2x}\cos 4x$ and $e^{-2x}\sin 4x$.

So putting our three sols together, we have $y = c_1e^x + c_2e^{-2x}\cos 4x + c_3e^{-2x}\sin 4x$

$$= c_1e^x + (c_2\cos 4x + c_3\sin 4x)e^{-2x}.$$

Problem 3 Find a **linear homogeneous constant coefficient** DEQ with the given general solution.

$$y(x) = (A + Bx + Cx^2)\cos 2x + (D + Ex + Fx^2)\sin 2x$$

Must've come from something like: $\cos 2x + i\sin 2x = e^{2ix}$ or $\cos 2x - i\sin 2x = e^{-2ix}$

$(r-2i), (r+2i)$ must be factors.

$(r-2i)(r+2i) = (r^2+4)$, What next?

$(r^2+4)^3 = r^6 + 12r^4 + 48r^2 + 64$, Are we done?

So the DEQ is: $y^{(6)} + 12y^{(4)} + 48y'' + 64y = 0$.

Problem 4 **DEQs with complex coefficients**.

Use the quadratic formula to solve the following. *Note in each case that the roots are not complex conjugates.*

a) $x^2 + ix + 2 = 0$

$$x = \frac{-i\pm\sqrt{-1-8}}{2} = -\frac{1}{2}i \pm \frac{3}{2}i = i \text{ or } -2i.$$

b) $x^2 - 2ix + 3 = 0$

$$x = \frac{2i\pm\sqrt{-4-12}}{2} = i \pm 2i = 3i \text{ or } i.$$