5.2 Exercises - Solutions 💪

Problem 1. Functions $y_1 = x$ and $y_2 = x^2$ are linearly independent solutions (on the entire real line) of the equation $x^2y'' - 2xy' + 2y = 0$. Verify that $W\left(x,\ x^2\right)$ vanishes at $x = 0$. Why do these observations not contradict part 2 of the Wronskian of Solutions Theorem?

To confirm linear independence, it is sufficient to note that you cannot represent x as $x = cx^2$, irrespective of what the constant c is. Next, create your Wronskian:

$$W\left(x,\ x^2\right) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2,$$ and we now see that the result vanishes at $x = 0$.

Finally, let's think about the Wronskian of Solutions Theorem: It assumes your equation has the form:

$$y'' + p_1(x)y' + p_2(x)y = 0,$$

where p_1, p_2 **are continuous** functions (on the interval of interest, near the initial condition).

However, if p_1, p_2 are NOT continuous functions there, we shouldn't expect the conclusions of the thm to hold.

When the equation $x^2y'' - 2xy' + 2y = 0$ is rewritten in the normal form: $y'' + (-\frac{2}{x})y' + \left(\frac{2}{x^2}\right)y = 0$ the coefficient functions $p_1(x) = -\frac{2}{x}$ and $p_2(x) = \frac{2}{x^2}$ are not continuous at $x = 0$. Thus, the assumptions of the theorem are not satisfied.

Problem 2. **Show that functions** $y_1(x) = e^x,\ y_2(x) = x^{-2},\ y_3(x) = x^{-2}\ln x$ **are linearly independent on** $(0, \infty)$.

$$W(x) = \begin{vmatrix} e^x & x^{-2} & x^{-2}\ln x \\ e^x & -2x^{-3} & x^{-3} - 2x^{-3}\ln x \\ e^x & 6x^{-4} & -5x^{-4} + 6x^{-4}\ln x \end{vmatrix} = e^x x^{-4} \begin{vmatrix} 1 & 1 & \ln x \\ 1 & -2x^{-1} & x^{-1} - 2x^{-1}\ln x \\ 1 & 6x^{-2} & -5x^{-2} + 6x^{-2}\ln x \end{vmatrix}$$

$$= e^x x^{-4} \begin{vmatrix} 1 & 1 & \ln x \\ 0 & -2x^{-1} - 1 & x^{-1} - 2x^{-1}\ln x - \ln x \\ 0 & 6x^{-2} - 1 & -5x^{-2} + 6x^{-2}\ln x - \ln x \end{vmatrix}$$

$$= e^x x^{-4}((-2x^{-1} - 1)(-5x^{-2} + 6x^{-2}\ln x - \ln x) - (x^{-1} - 2x^{-1}\ln x - \ln x)(6x^{-2} - 1))$$

$$= e^x x^{-4}((10x^{-3} - 12x^{-3}\ln x + 2x^{-1}\ln x) + (5x^{-2} - 6x^{-2}\ln x + \ln x) + (-6x^{-3} + 12x^{-3}\ln x + 6x^{-2}\ln x) + (x^{-1} - 2x^{-1}\ln x - \ln x$$

$$= e^x x^{-4}((10x^{-3} - 12x^{-3}\ln x - 6x^{-3} + 12x^{-3}\ln x) + (5x^{-2} - 6x^{-2}\ln x + 6x^{-2}\ln x) + (2x^{-1}\ln x + x^{-1} - 2x^{-1}\ln x) + (\ln x - \ln x)$$

$$= e^x x^{-4}(4x^{-3} + 5x^{-2} + x^{-1}),$$ which is nonzero on $(0, \infty)$.

Therefore, the functions are linearly independent.

Problem 3 **Use the Wronskian to prove that the functions** $\{x, \cos(\ln x),\ \sin(\ln x)\}$ **are linearly independent on the interval** $x > 0$.

$$W(x) = \begin{vmatrix} x & \cos(\ln x) & \sin(\ln x) \\ 1 & -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \\ 0 & -\frac{\frac{1}{x}\cos(\ln x)(x) - \sin(\ln x)}{x^2} & \frac{-\frac{1}{x}\sin(\ln x)(x) - \cos(\ln x)}{x^2} \end{vmatrix} = \begin{vmatrix} x & \cos(\ln x) & \sin(\ln x) \\ 1 & -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \\ 0 & \frac{-\cos(\ln x) + \sin(\ln x)}{x^2} & \frac{-\sin(\ln x) - \cos(\ln x)}{x^2} \end{vmatrix}$$

$$= x\left(-\frac{\sin(\ln x)}{x}\frac{-\sin(\ln x) - \cos(\ln x)}{x^2} - \frac{\cos(\ln x)}{x}\frac{-\cos(\ln x) + \sin(\ln x)}{x^2}\right) - \left(\cos(\ln x)\frac{-\sin(\ln x) - \cos(\ln x)}{x^2} - \sin(\ln x)\frac{-\cos(\ln x) + \sin(\ln x)}{x^2}\right)$$

$$= \frac{\sin^2(\ln x) + \sin(\ln x)\cos(\ln x)}{x^2} - \frac{-\cos^2(\ln x) + \sin(\ln x)\cos(\ln x)}{x^2} + \frac{\cos(\ln x)\sin(\ln x) + \cos^2(\ln x)}{x^2} + \frac{-\sin(\ln x)\cos(\ln x) + \sin^2(\ln x)}{x^2}.$$

So, $W(x) = x^{-2}[2\cos^2(\ln x) + 2\sin^2(\ln x)] = 2x^{-2}$.

And, $W(x)$ is nonzero (and defined) for $x > 0$.

So, the functions $\{x, \cos(\ln x),\ \sin(\ln x)\}$ are linearly independent on the interval $x > 0$.