4.3 Exercises - Solutions

Problem 1 If possible, express $\vec{w} = (7,7,9,11)$ as a linear combination of
$\vec{v}_1 = (2,0,3,1),\ \vec{v}_2 = (4,1,3,2),\ \vec{v}_3 = (1,3,-1,3).$ If not, show that it is impossible.

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w}$ $\qquad c_1(2,0,3,1) + c_2(4,1,3,2) + c_3(1,3,-1,3) = (7,7,9,11)$

$$\mathbf{A}\vec{c} = \vec{w} \Rightarrow \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 3 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 9 \\ 11 \end{bmatrix} \overset{\text{trust me}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

Has the unique solution…

$c_1 = 6,\ c_2 = -2,\ c_3 = 3,$ so...

$\vec{w} = 6\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3.$

Want to be sure you got the right answer? Substitute into this equation the relevant vectors to ensure you get $\vec{w} = (7,7,9,11)$.

Problem 2 If the following vectors are linearly independent, show it.
Otherwise, find a nontrivial linear combination of them that is equal to the zero vector.
$\vec{v}_1 = (3,9,0,5),\ \vec{v}_2 = (3,0,9,-7),\ \vec{v}_3 = (4,7,5,0)$

$$\mathbf{A} = \begin{bmatrix} 3 & 3 & 4 \\ 9 & 0 & 7 \\ 0 & 9 & 5 \\ 5 & -7 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & \frac{5}{9} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that the system of 4 eqns in 3 unknowns has a 1D soln space.

$c_3 = s, \qquad c_1 = -\frac{7}{9}s, \qquad c_2 = -\frac{5}{9}s$

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{9}s \\ -\frac{5}{9}s \\ s \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -9 \end{bmatrix}, \text{ when } s = -9.$$

(Since s is a parameter, and can therefore be any real number, I have chosen -9 as its value for notational convenience.)

So, we have $c_1 = 7,\ c_2 = 5,$ and $c_3 = -9.$

Therefore $7\vec{v}_1 + 5\vec{v}_2 - 9\vec{v}_3 = \vec{0}.$

(on a test, you will want to double check this by making sure the equality holds by plugging in the vectors)

Problem 3 Let's assume the set of vectors $\{\vec{v}_i\}$ are linearly independent. Apply the definition of linear independence to show that the following vectors are also linearly independent.
$\vec{u}_1 = \vec{v}_2 + \vec{v}_3, \quad \vec{u}_2 = \vec{v}_1 + \vec{v}_3, \quad \vec{u}_3 = \vec{v}_1 + \vec{v}_2.$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, has only the trivial solution $c_1 = c_2 = c_3 = 0$.

Want to show that $b_1\vec{u}_1 + b_2\vec{u}_2 + b_3\vec{u}_3 = \vec{0}$, has only the trivial solution $b_1 = b_2 = b_3 = 0$.

$(*) \quad b_1\vec{u}_1 + b_2\vec{u}_2 + b_3\vec{u}_3 = b_1(\vec{v}_2 + \vec{v}_3) + b_2(\vec{v}_1 + \vec{v}_3) + b_3(\vec{v}_1 + \vec{v}_2)$

$= (b_2 + b_3)\vec{v}_1 + (b_1 + b_3)\vec{v}_2 + (b_1 + b_2)\vec{v}_3.$

Setting this equal to zero, by our previous assumption it must be that $b_2 + b_3 = 0, \quad b_1 + b_3 = 0$, and $b_1 + b_2 = 0$.

From the first equation we have: $b_3 = -b_2$.

Applying this to the second equation, we have: $b_1 = b_2$.

And then from the third equation,we get: $2b_2 = 0$ or $b_2 = 0$. But then $b_1 = 0$, and $b_3 = 0$.

Therefore, only the trivial solution satisfies the equation $(*)$, and the vectors $\{\vec{u}_i\}$ are therefore linearly independent.

Problem 4 **Prove**: If a set S of vectors is linearly dependent and a (finite) set T contains S, then T is also linearly dependent. Assume $S = \{\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_k\}$ and $T = \{\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_k, \ldots, \vec{v}_m\}$, with $m > k$.

We need to find nontrivial $c_1, c_2, \ldots, c_m$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m = \vec{0}$.

Because the set S of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is linearly dependent,

there exist scalars $c_1, c_2, \ldots, c_k$ not all zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$.

Now let $c_{k+1} = \ldots = c_m = 0$.

So we have: $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \ldots + c_m\vec{v}_m = \vec{0}$ with the coefficients $c_1, c_2, \ldots, c_m$ not all zero.

This means that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ that define T are linearly dependent.