

4.2 Exercises - Solutions

Problem 1 For the following system of equations, find two solution vectors \vec{u} and \vec{v} (a basis) such that the **solution space** is the set of all linear combinations of the form $s\vec{u} + t\vec{v}$.

$$\begin{aligned}x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\2x_1 - x_2 + x_3 + 7x_4 &= 0 \\x_1 + 2x_2 + 3x_3 + 11x_4 &= 0\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \xrightarrow{\text{trust me}} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $x_3 = s$ and $x_4 = t$ are free variables. We solve for $x_2 = -s - 3t$, and $x_1 = -s - 5t$. So ...

$$\vec{x} = (x_1, x_2, x_3, x_4) = (-s - 5t, -s - 3t, s, t)$$

$$= (-s, -s, s, 0) + (-5t, -3t, 0, t) = s\vec{u} + t\vec{v}, \text{ where } \vec{u} = (-1, -1, 1, 0) \text{ and } \vec{v} = (-5, -3, 0, 1).$$

Problem 2 Reduce the given system to echelon form to find a single solution vector \vec{u} such that the solution space is the set of all scalar multiples of \vec{u} .

$$\begin{aligned}x_1 + 3x_2 + 3x_3 + 3x_4 &= 0, \\2x_1 + 7x_2 + 5x_3 - x_4 &= 0, \\2x_1 + 7x_2 + 4x_3 - 4x_4 &= 0.\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 7 & 5 & -1 \\ 2 & 7 & 4 & -4 \end{bmatrix} \xrightarrow{\text{trust me}} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus $x_4 = t$ is a parameter (a.k.a. free variable).

We solve for $x_1 = -6t$, $x_2 = 4t$, and $x_3 = -3t$. So,

$$\vec{x} = (x_1, x_2, x_3, x_4) = (-6t, 4t, -3t, t) = t\vec{u}, \text{ where } \vec{u} = (-6, 4, -3, 1).$$

Problem 3 Let \mathbf{A} be an $n \times n$ matrix, \vec{b} be a nonzero vector, and \vec{x}_0 be a solution vector to the system $\mathbf{A}\vec{x} = \vec{b}$. Show that \vec{x}_2 is another solution **if and only if** $(\Leftrightarrow) \vec{x}_2 - \vec{x}_0$ is a solution of the homogeneous system $\mathbf{A}\vec{y} = \vec{0}$.

We are given: $\mathbf{A}\vec{x}_0 = \vec{b}$.

Need to show that: $\mathbf{A}\vec{x}_2 = \vec{b} \Leftrightarrow \mathbf{A}(\vec{x}_2 - \vec{x}_0) = \vec{0}$.

Starting with the left assumption, and trying to show the statement on the right, we have:

$$\mathbf{A}(\vec{x}_2 - \vec{x}_0) = \mathbf{A}\vec{x}_2 - \mathbf{A}\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}. \quad \checkmark$$

Going from right to left, we have: $\mathbf{A}(\vec{x}_2 - \vec{x}_0) = \mathbf{A}\vec{x}_2 - \mathbf{A}\vec{x}_0 = \mathbf{A}\vec{x}_2 - \vec{b} = \vec{0}$, therefore $\mathbf{A}\vec{x}_2 = \vec{b}$. **Q.E.D.**

Problem 4. Assume W is the set of all vectors in \mathbb{R}^4 such that $x_1 = 3x_3$ and $x_2 = 4x_4$. Apply the theorems in this section to determine whether or not W is a subspace of \mathbb{R}^4 .

$$W = \{(3c, 4d, c, d)\}.$$

First, note that the subspace is nonempty since $(3, 4, 1, 1) \in W$, where $c, d = 1$.

We arbitrarily choose two vectors from W by arbitrarily choosing four constant $c_1, d_1, c_2, d_2 \in \mathbb{R}$, giving us $(3c_1, 4d_1, c_1, d_1)$ and $(3c_2, 4d_2, c_2, d_2)$. We then test them for closure under addition:

$$\begin{aligned} &(3c_1, 4d_1, c_1, d_1) + (3c_2, 4d_2, c_2, d_2) \\ &= (3c_1 + 3c_2, 4d_1 + 4d_2, c_1 + c_2, d_1 + d_2) \end{aligned}$$

$$= (3(c_1 + c_2), 4(d_1 + d_2), c_1 + c_2, d_1 + d_2) \in W$$

This is because it has the prescribed format $\{(3c, 4d, c, d)\}$, where $c = c_1 + c_2$ and $d = d_1 + d_2$.

Now to test scalar multiplication:

$$\alpha(3c_1, 4d_1, c_1, d_1) = (3\alpha c_1, 4\alpha d_1, \alpha c_1, \alpha d_1) \in W = \{(3c, 4d, c, d)\}$$

where $c = \alpha c_1$ and $d = \alpha d_1$.

Therefore, W is nonempty, closed under addition, and scalar multiplication, and is a subspace of \mathbb{R}^4 . ■