4.1 Exercises - Solutions

Problem 1 Determine whether the given vectors $\vec{u} = (1,4,5)$, $\vec{v} = (4,2,5)$, $\vec{w} = (-3,3,-1)$ are linearly independent or dependent. If they are linearly dependent, find scalars $a, b,$ and c not all zero such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$.

So, $\mathbf{A}\vec{z} = 0$, where $\vec{z} = \begin{bmatrix} a & b & c \end{bmatrix}^T$, and $\mathbf{A} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} 1 & 4 & -3 \\ 4 & 2 & 3 \\ 5 & 5 & -1 \end{bmatrix}$.

$$\mathbf{A} \Rightarrow \begin{bmatrix} 1 & 4 & -3 \\ 0 & -14 & 15 \\ 0 & -15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & -15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 29 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system $\mathbf{Ax} = \vec{\mathbf{0}}$ has only the trivial solution $a = b = c = 0$ ($\mathbf{A}$ is invertible), so the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent.

Problem 2 Show that V, defined as the set of all (x,y,z) such that $z = 2x + 3y$, is closed under addition and under multiplication by scalars, and is therefore a subspace of $\mathbb{R}^3$.

If one were to choose $\vec{u}$ and $\vec{v}$ randomly from V and choose $c \in \mathbb{R}$, we would then need to show that $\vec{u} + \vec{v}$ and $c\vec{v}$ are members of V.

$\vec{u} = \left(u_1,\ u_2,\ 2u_1 + 3u_2\right)$ and $\vec{v} = \left(v_1,\ v_2,\ 2v_1 + 3v_2\right)$ for some $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

Closed Under Addition?

$\vec{u} + \vec{v} = \left(u_1 + v_1,\ u_2 + v_2,\ (2u_1 + 3u_2) + (2v_1 + 3v_2)\right)$

$= \left(u_1 + v_1,\ u_2 + v_2,\ 2(u_1 + v_1) + 3(u_2 + v_2)\right)$ where $u_1 + v_1,\ u_2 + v_2 \in \mathbb{R}$.

$\vec{u} + \vec{v}$ is in the form (x,y,z) such that $z = 2x + 3y$, so $\vec{u} + \vec{v} \in V$. ✓

Closed Under Scalar Multiplication?

$c\vec{v} = \left(cv_1,\ cv_2,\ 2cv_1 + 3cv_2\right)$ where $cv_1,\ cv_2,\ 2cv_2 + 3cv_2 \in \mathbb{R}$.

$c\vec{v}$ is also in the form (x,y,z) such that $z = 2x + 3y$, so $c\vec{v} \in V$. ✓

Problem 3 Show that V, the set of all (x,y,z) such that $y = 1$, is not a subspace of $\mathbb{R}^3$.

Just need an example of vector(s) in V which are not closed under scalar multiplication or under addition.

$(0,1,0)$ is in V, but the sum $(0,1,0) + (0,1,0) = (0,2,0)$ is not in V.

Thus V is not closed under addition, so not a subspace of $\mathbb{R}^3$.