3.6 Exercises - Solutions 💪

Problem 1 Use the method of elimination (*determinant* row/column operations) to evaluate: $\begin{vmatrix} -2 & 5 & 4 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix}$

$$\Rightarrow \quad c_2 + (-4c_1) \quad \Rightarrow \begin{vmatrix} -2 & 13 & 4 \\ 5 & -17 & 1 \\ 1 & 0 & 5 \end{vmatrix} \quad \Rightarrow \quad c_3 + (-5c_1) \quad \Rightarrow \begin{vmatrix} -2 & 13 & 14 \\ 5 & -17 & -24 \\ 1 & 0 & 0 \end{vmatrix}$$

(note that the above operations can only be performed on determinants, not matrices!)

$$\Rightarrow \quad -\begin{vmatrix} -2 & 13 & 14 \\ 1 & 0 & 0 \\ 5 & -17 & -24 \end{vmatrix} \quad \Rightarrow \quad +\begin{vmatrix} 1 & 0 & 0 \\ -2 & 13 & 14 \\ 5 & -17 & -24 \end{vmatrix} = +1\begin{vmatrix} 13 & 14 \\ -17 & -24 \end{vmatrix} = -74. \quad \text{(Checkerboard and Fish)}$$

(don't need these steps here, but thought I'd show row-swapping/sign-changes)

Problem 2 Use Cramer's rule to solve the system:

$$\begin{aligned} 2x_1 - 5x_3 &= -3 \\ 4x_1 + 3x_3 - 5x_2 &= 3 \\ -2x_1 + x_2 + x_3 &= 1 \end{aligned}$$

$$\mathbf{A}\vec{x} = \vec{b} \qquad |\mathbf{A}| = \begin{vmatrix} 2 & 0 & -5 \\ 4 & -5 & 3 \\ -2 & 1 & 1 \end{vmatrix} \quad \text{(let's further simplify the column with the zero in it)}$$

$$|\mathbf{A}| = \begin{vmatrix} 2 & 0 & -5 \\ -6 & 0 & 8 \\ -2 & 1 & 1 \end{vmatrix} = -1(16-30) = 14 \neq 0. \quad \checkmark$$

$$x_1 = \frac{|\mathbf{B}_1|}{|\mathbf{A}|} = \frac{1}{14}\begin{vmatrix} \mathbf{-3} & 0 & -5 \\ \mathbf{3} & -5 & 3 \\ \mathbf{1} & 1 & 1 \end{vmatrix} = -\frac{8}{7}, \quad x_2 = \frac{|\mathbf{B}_2|}{|\mathbf{A}|} = \frac{1}{14}\begin{vmatrix} 2 & \mathbf{-3} & -5 \\ 4 & \mathbf{3} & 3 \\ -2 & \mathbf{1} & 1 \end{vmatrix} = -\frac{10}{7}, \quad x_3 = \frac{1}{14}\begin{vmatrix} 2 & 0 & \mathbf{-3} \\ 4 & -5 & \mathbf{3} \\ -2 & 1 & \mathbf{1} \end{vmatrix} = \frac{1}{7}$$

$$\text{So,} \quad \vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \frac{1}{7}\begin{bmatrix} -8 \\ -10 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -8 & -10 & 1 \end{bmatrix}^T.$$

Problem 3 Use the cofactor method of this section to find the inverse ($\mathbf{A}^{-1}$) of:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -3 \\ 2 & -3 & -1 \\ -5 & 0 & -3 \end{bmatrix}.$$ Then use it to solve $\mathbf{A}\vec{x} = \vec{b}$ where $\vec{b} = [0,1,1]^T$. (HINT $\det \mathbf{A} = 107$)

Recall: $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T$. **Cofactors**: $A_{mn} = $ (Checkerboard)(Fish).

So, $A_{11} = (+)(9-0) = 9$, $A_{12} = (-)(-6-5) = 11$, $A_{13} = (+)(0-15)\ldots$ So, $[A_{mn}] = \begin{bmatrix} 9 & 11 & -15 \\ 12 & -21 & -20 \\ -13 & -4 & -14 \end{bmatrix}$.

And the transpose: $[A_{mn}]^T$ is $\begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}$, so $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T = \frac{1}{107}\begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}$.

Solution: $\vec{x} = \frac{1}{107}\begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{107}\begin{bmatrix} -1 \\ -25 \\ -34 \end{bmatrix}$.

Problem 4 Show that: $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4$, and $\begin{vmatrix} 2 & 1 & 0 & 9 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{vmatrix} = 1$

Top row: $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2(2\cdot 2 - 1\cdot 1) - 1(1\cdot 2 - 1\cdot 0) + 0 = 6 - 2 = 4.$ OR…

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \overset{c_3+(-2c_2)}{=} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 2 & -3 \\ 0 & 1 & 0 \end{vmatrix} = 0 + (-)[2\cdot(-3) - (-2\cdot 1)] + 0 = 4,$$ (where I use the last row).

For the second determinant: $$\begin{vmatrix} 2 & 1 & 0 & 9 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{vmatrix} \overset{c_3+(-2c_2)}{=} \begin{vmatrix} 2 & 1 & -2 & 9 \\ 1 & 2 & -2 & 3 \\ 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix} = +1\begin{vmatrix} 2 & -2 & 9 \\ 1 & -2 & 3 \\ 1 & -3 & 1 \end{vmatrix} \overset{c_2+3c_1}{=} \begin{vmatrix} 2 & 4 & 9 \\ 1 & 1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$$

$$\overset{c_3+(-c_1)}{=} \begin{vmatrix} 2 & 4 & 7 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 7 \\ 1 & 2 \end{vmatrix} = 8 - 7 = 1.$$ Or just powering through on the first row...

$$\begin{vmatrix} 2 & 1 & 0 & 9 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{vmatrix} = 2A_{11} + A_{12} + 0 \cdot A_{13} + 9A_{14} = 2\begin{vmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} - 9\begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 2(1 \cdot (2-3) - 2(2-6)) - (-2)(1-3) - 9(1 \cdot (4-1) - 1 \cdot (4-2)) = 2(7) - 4 - 9(1) = 1.$$