3.5 Exercises - Solutions 💪

Problem 1 Find $\mathbf{A}^{-1}$, then use $\mathbf{A}^{-1}$ to solve the system $\mathbf{A}\vec{x} = \vec{b}$. $\mathbf{A} = \begin{bmatrix} 7 & 9 \\ 5 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\mathbf{A}^{-1} = -\frac{1}{10}\begin{bmatrix} 5 & -9 \\ -5 & 7 \end{bmatrix};$ $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

$\vec{x} = -\frac{1}{10}\begin{bmatrix} 5 & -9 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}.$

Problem 2 Use the identity matrix to find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{R2+(-3R1)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R3+(-R1)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \overset{R2+(-5R3)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 1 & -5 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{R3+(-R2)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 1 & -5 \\ 0 & 0 & 5 & -3 & -1 & 6 \end{array}\right] \overset{R2+R3}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 5 & -3 & -1 & 6 \end{array}\right]$$

$$\overset{R1+2R3}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 5 & -3 & -1 & 6 \end{array}\right] \overset{\frac{1}{5}R3}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & \frac{6}{5} \end{array}\right]$$

(notice how I am avoiding fractions until the last possible moment)

$$\overset{R1+(-2R2)}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & \frac{6}{5} \end{array}\right]. \quad \text{So, } \mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & -2 \\ -5 & 0 & 5 \\ -3 & -1 & 6 \end{bmatrix}.$$

Problem 3 Find a matrix **X** (matrix FULL of unknowns!!!) such that $\mathbf{AX} = \mathbf{B}$,

where $\mathbf{A} = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & -2 \\ 1 & 7 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Since $\mathbf{AX} = \mathbf{B}$, If we knew that **A** had an inverse, we could multiply both sides of the eqn (on the left) by $\mathbf{A}^{-1}$ and get:

$(\mathbf{A}^{-1})\mathbf{AX} = (\mathbf{A}^{-1})\mathbf{B}$.

The left hand side becomes: $(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$. So, calculate $\mathbf{A}^{-1} = \begin{bmatrix} -16 & 3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix}$;

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -16 & 3 & 11 \\ 6 & -1 & -4 \\ -13 & 2 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -21 & 9 & 6 \\ 8 & -3 & -2 \\ -17 & 6 & 5 \end{bmatrix}.$$

Problem 4. Let **A** be an $n \times n$ matrix with either a row or a column consisting only of zeros. Show that **A** is not invertible.

$$\mathbf{A}\vec{x} = \begin{bmatrix} a'_{11} & a'_{12} & \dots & a'_{1n} \\ 0 & 0 & 0 & 0 \\ a'_{31} & a'_{32} & \dots & a'_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \dots & a'_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}.$$

$\mathbf{A}\vec{x} = \mathbf{0}$ gives us $n-1$ eqns (observe the 2nd eqn is not an information giving eqn) in n unknowns $\{x_1, \dots, x_n\}$, so infinitely many sols. However, recall that for invertible matrices: "$\mathbf{A}\vec{x} = \mathbf{0}$ has only the trivial soln."

How about: $$\mathbf{A}\vec{x} = \begin{bmatrix} a_{11} & 0 & a_{13} & \dots & a_{1n} \\ a_{21} & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & a_{n3} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}. \quad ?$$

Recall that for invertible matrices, we had: "$\mathbf{A}\vec{x} = \vec{0}$ has only the trivial soln." However, observe that:

$\mathbf{A}\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = x_1\vec{a}_1 + x_2\vec{0} + \dots + x_n\vec{a}_n = \vec{0}$. But this means that x_2 can be anything and still satisfy the above eqn. So, the trivial solution is not the only one.

(actually, infinitely many more)