3.3 Exercises - Solutions 💪

Problem 1 Find the reduced echelon form of the following matrix $\begin{bmatrix} 1 & 3 & 15 & 7 \\ 2 & 4 & 22 & 8 \\ 2 & 7 & 34 & 17 \end{bmatrix}$.

$$\overset{R2+(-2R1)}{\rightarrow} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & -2 & -8 & -6 \\ 2 & 7 & 34 & 17 \end{bmatrix} \overset{R3+(-2R1)}{\rightarrow} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & -2 & -8 & -6 \\ 0 & 1 & 4 & 3 \end{bmatrix} \overset{-\frac{1}{2}R2}{\rightarrow} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

$$\overset{R3+(-R2)}{\rightarrow} \begin{bmatrix} 1 & 3 & 15 & 7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is echelon form.

OPTIONAL: Assuming this matrix was augmented, use back substitution to solve:

$$\overset{R1+(-3R2)}{\rightarrow} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, $x_3 = t_3, \quad x_2 = -4t_3 + 3, \quad x_1 = -3t_3 - 2.$

Vector form: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3t_3 - 2 \\ -4t_3 + 3 \\ t_3 \end{bmatrix} = \begin{bmatrix} -3t_3 \\ -4t_3 \\ t_3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} = t_3 \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}.$

Problem 2 Use the method of Gauss-Jordan elimination (reduced echelon form) to determine how many solutions the system has.

$$\begin{bmatrix} 2 & 2 & 4 & 2 \\ 1 & -1 & -4 & 3 \\ 2 & 7 & 19 & 7 \end{bmatrix}$$

$$\Rightarrow \tfrac{1}{2}R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & -4 & 3 \\ 2 & 7 & 19 & 7 \end{bmatrix} \Rightarrow R_3 + (-2R_1) \text{ and } R_2 + (-R_1) \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -6 & 2 \\ 0 & 5 & 15 & 5 \end{bmatrix}$$

$$\Rightarrow -\tfrac{1}{2}R_2 \text{ and } \tfrac{1}{5}R_3 \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \Rightarrow R_3 + (-R_2) \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow R_1 + (-R_2) \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \quad 0 \stackrel{?}{=} 2 \qquad (!?!)$$

How many solutions? No solutions.

Problem 3 Consider a homogeneous system (what's this?) of three equations in three unknowns. Suppose the third equation is the sum of a multiple of the first equation and a multiple of the second equation. Show that the system has a nontrivial solution (what's this?).

It is given that the augmented coefficient matrix of the homogeneous 3 x 3 system has the form…

$$\begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ pa_1 + qa_2 & pb_1 + qb_2 & pc_1 + qc_2 & 0 \end{bmatrix}.$$

Upon subtracting both p times row 1 and q times row 2 from row 3, we get the matrix…

$$\Rightarrow R_3 + (-pR_1) \text{ and } R_3 + (-qR_2) \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

... corresponding to two homogeneous linear equations in three unknowns. Hence there is at least one free variable, and thus the system has a nontrivial family of (infinitely many) solutions.

Problem 4 **Show that the** 2×2 **matrix**: $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **is row equivalent to the** 2×2 **identity matrix**, **provided that** $ad - bc \neq 0$. Observe that the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

HINTS: For them to be row equivalent, we need to be able to transform one into the other.

Try to do row operations on **A**, to get it into some form which looks like **I**.

Observe what restrictions on a, b, c, d arise as you do this (do not divide by zero!).

$$\stackrel{\frac{1}{a}R_1}{\Rightarrow} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \stackrel{R_2+(-cR_1)}{\Rightarrow} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix} \stackrel{\frac{a}{ad-bc}R_2}{\Rightarrow} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} \stackrel{R_1+\left(-\frac{b}{a}R_2\right)}{\Rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Observe that we needed $ad - bc \neq 0$ in the 2nd to last step, but we also needed $a \neq 0$ in the first step.

However, observe that we could have switched the rows for our first step. If we had done this, the calculations would have been nearly identical, except we would have required that $c \neq 0$. So we need that either a or c not equal zero. Observe that $ad - bc \neq 0$ requires that either a or c not equal zero, so indeed this one requirement is sufficient to show that **A** is row equivalent to I.