3.2 Exercises - Solutions 💪

Problem 1 This linear system is in **echelon form**. Solve it by back substitution.

$$\begin{aligned} x_1 - 10x_2 + 3x_3 - 13x_4 &= 5 \\ x_3 + 3x_4 &= 10 \end{aligned}$$

If we set $x_2 = s$ and $x_4 = t$, then the second equation gives $x_3 = 10 - 3t$, and the first equation gives …

$x_1 - 10s + 3(10 - 3t) - 13t = 5.$ So, $x_1 = 10s - 3(10 - 3t) + 13t + 5 = -25 + 10s + 22t.$

So the infinite solution set is: $(x_1, x_2, x_3, x_4) = \left(-25 + 10s + 22t,\ s,\ 10 - 3t,\ t\right)$, for every $s, t \in \mathbb{R}$.

Problem 2 Use elementary row operations to transform the following in a coefficient matrix to **echelon form**. Then solve the system by back substitution.

$$\begin{aligned} 4x_1 - 2x_2 - 3x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 - 5x_3 &= -10 \\ 4x_1 + x_2 + 2x_3 + x_4 &= 17 \\ 3x_1 + x_3 + x_4 &= 12 \end{aligned}$$

Turn the system into: $\left[\begin{array}{cccc|c} 4 & -2 & -3 & 1 & 3 \\ 2 & -2 & -5 & 0 & -10 \\ 4 & 1 & 2 & 1 & 17 \\ 3 & 0 & 1 & 1 & 12 \end{array}\right] \Rightarrow R_1 + (-R_4) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 2 & -2 & -5 & 0 & -10 \\ 4 & 1 & 2 & 1 & 17 \\ 3 & 0 & 1 & 1 & 12 \end{array}\right]$

$\Rightarrow R_2 + (-2R_1) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 2 & 3 & 0 & 8 \\ 4 & 1 & 2 & 1 & 17 \\ 3 & 0 & 1 & 1 & 12 \end{array}\right] \Rightarrow R_3 + (-4R_1) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 2 & 3 & 0 & 8 \\ 0 & 9 & 18 & 1 & 53 \\ 3 & 0 & 1 & 1 & 12 \end{array}\right]$

$\Rightarrow R_4 + (-3R_1) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 2 & 3 & 0 & 8 \\ 0 & 9 & 18 & 1 & 53 \\ 0 & 6 & 13 & 1 & 39 \end{array}\right] \Rightarrow R_2 \leftrightarrow R_3 \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 9 & 18 & 1 & 53 \\ 0 & 2 & 3 & 0 & 8 \\ 0 & 6 & 13 & 1 & 39 \end{array}\right]$

$\Rightarrow R_3 + (-4R_3) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 1 & 6 & 1 & 21 \\ 0 & 2 & 3 & 0 & 8 \\ 0 & 6 & 13 & 1 & 39 \end{array}\right] \Rightarrow R_3 + (-2R_3) \text{ and } R_4 + (-6R_2) \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -4 & 0 & -9 \\ 0 & 1 & 6 & 1 & 21 \\ 0 & 0 & -9 & -2 & -34 \\ 0 & 0 & -23 & -5 & -87 \end{array}\right]$

$$\Rightarrow R_4 + (-2R_3) \Rightarrow \begin{bmatrix} 1 & -2 & -4 & 0 & | & -9 \\ 0 & 1 & 6 & 1 & | & 21 \\ 0 & 0 & -9 & -2 & | & -34 \\ 0 & 0 & -5 & -1 & | & -19 \end{bmatrix} \Rightarrow R_3 + (-2R_4) \Rightarrow \begin{bmatrix} 1 & -2 & -4 & 0 & | & -9 \\ 0 & 1 & 6 & 1 & | & 21 \\ 0 & 0 & 1 & 0 & | & 4 \\ 0 & 0 & -5 & -1 & | & -19 \end{bmatrix}$$

$$\Rightarrow R_4 + 5R_3 \Rightarrow \begin{bmatrix} 1 & -2 & -4 & 0 & | & -9 \\ 0 & 1 & 6 & 1 & | & 21 \\ 0 & 0 & 1 & 0 & | & 4 \\ 0 & 0 & 0 & -1 & | & 1 \end{bmatrix}.$$ **What is next**?

$x_4 = -1, \quad x_3 = 4, \quad x_2 = -6x_3 - x_4 + 21 \; = \; -6(4) - (-1) + 21 \; = -2$

$x_1 = 2x_2 + 4x_3 - 9 \; = \; 2(-2) + 4(4) - 9 \; = 3.$ The (only) solution is: $\vec{x} = (3, -2, \; 4, -1)$.

Problem 3 Determine for what values of k the following system has

a) a unique solution: *b*) no solution; *c*) infinitely many solutions.

$$3x + 2y = 1$$
$$7x + 5y = k$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 7 & 5 & k \end{bmatrix}$$ (Note, I left the " | " column out this time)

$$\Rightarrow R_2 + (-2R_1) \Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & k-2 \end{bmatrix} \Rightarrow R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} 1 & 1 & k-2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow R_2 + (-3R_1) \Rightarrow \begin{bmatrix} 1 & 1 & k-2 \\ 0 & -1 & -3k+7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & k-2 \\ 0 & 1 & 3k-7 \end{bmatrix}.$$

Unique solution for any k. $\quad y = 3k - 7, \quad x = -y + (k-2) = -(3k-7) + k - 2 = -2k + 5,$

$\vec{v} = \begin{bmatrix} -2k+5 & 3k-7 \end{bmatrix}.$

"**For what values of k does the system have**: **a**) **a unique solution**: **b**) **no solution**; **c**) **infinitely many solutions**."

Since no free variables (and no $0 \cdot x + 0 \cdot y = c$ equations), it follows that the given system has a unique solution for every the value of k.

If our system, at the end, had instead looked like: $\begin{bmatrix} 1 & 1 & k-2 \\ 0 & 0 & 0 \end{bmatrix}$, we would have concluded that

there was an infinite number of solutions.

If our system, at the end, had looked like: $\begin{bmatrix} 1 & 1 & k-2 \\ 0 & 0 & 3k-7 \end{bmatrix}$, we would have concluded that

there were no solutions, except for when $k = \frac{7}{3}$, in which case there would be an infinite number of solutions.