2.4 Exercises - Solutions

Problem 1 Consider the initial value problem: $y' = e^{-y}, \quad y(0) = 0$,

with exact solution: $y(x) = \ln(x+1)$.

Apply Euler's method to approximate to this solution on the interval $\left[0, \frac{1}{2}\right]$;

first with step size $h_1 = 0.25$, then with step size $h_2 = 0.1$.

Compare the values of the approximations at $x = \frac{1}{2}$

with the value $y\left(\frac{1}{2}\right)$ of the actual solution ($y(\frac{1}{2}) = \ln\left(\frac{1}{2}+1\right) \approx 0.405$).

Recall: Iterative formula: $y_{n+1} = y_n + h \cdot e^{-y_n}$.

$y_1 = 0 + (0.25)e^{-0} = 0.25$

$y_2 = 0.25 + (0.25)e^{-0.25} \approx 0.4447$.

$y_1 = 0 + (0.1)e^{-0} = 0.1$

$y_2 = 0.1 + (0.1)e^{-0.1} = 0.19048$

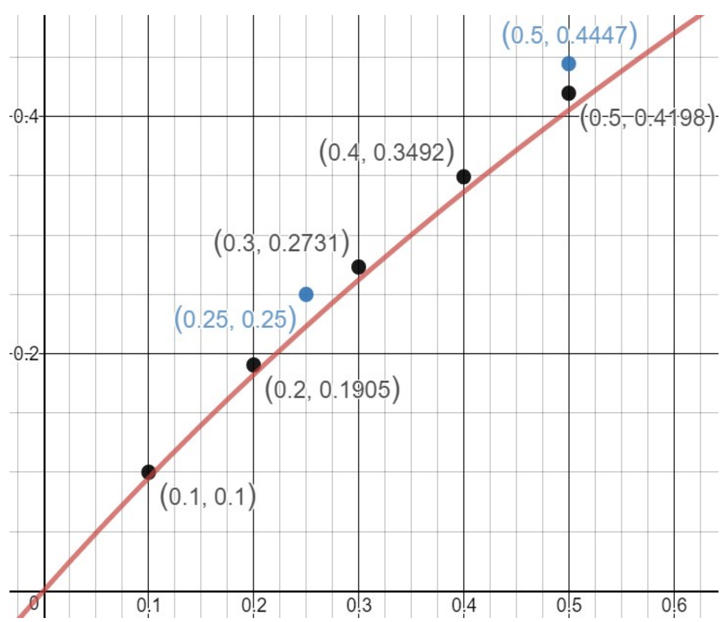
$y_3 = 0.19048 + (0.1)e^{-0.19048} = 0.27314$

$y_4 = 0.27314 + (0.1)e^{-0.27314} = 0.34924$

$y_5 = 0.34924 + (0.1)e^{-0.34924} = 0.41976$

Approximate values: $h_1 : 0.445$ and $h_2 : 0.420$;

True value? $y\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}+1\right) \approx 0.405$.



Problem 2 Consider the initial value problem: $y' = 2xy^2,\ y(0) = 1$; Exact solution: $y(x) = \frac{1}{1-x^2}$.

Apply Euler's method to approximate to this solution on the interval $\left[0, \frac{1}{2}\right]$;

First with step size $h_1 = 0.25$, then with step size $h_2 = 0.1$. Compare with the exact solution above.

Iterative formula: $y_{n+1} = y_n + h(2x_n y_n^2)$

$y_1 = 1 + (0.25)(2 \cdot 0 \cdot 1^2) = 1,$

$y_2 = 1 + (0.25)(2 \cdot (0.25) \cdot 1^2) = 1.125.$

$y_1 = 1 + (0.1)(2 \cdot (0) \cdot 1^2) = 1,$

$y_2 = 1 + (0.1)(2 \cdot (0.1) \cdot 1^2) = 1.02,$

$y_3 = (1.02) + (0.1)\left(2 \cdot (0.2) \cdot (1.02)^2\right) = 1.0616,$

$y_4 = (1.0616) + (0.1)\left(2 \cdot (0.3) \cdot (1.0616)^2\right) = 1.1292,$

$y_5 = (1.1292) + (0.1)\left(2 \cdot (0.4) \cdot (1.1292)^2\right) = 1.2312.$

Approximate values: $h_1 = 1.125$ and $h_2 = 1.231$;

True value: $y\left(\frac{1}{2}\right) \approx 1.333.$