2.2 Exercises - Solutions

Problem 1. **Which of the following is true about the solution of the initial value problem**:

$\frac{dx}{dt} = 6x - x^2 - 8, \; x(0) = 1$

$A)$ $\lim_{t\to\infty} x(t) = -2$ $\quad B)$ $\lim_{t\to-\infty} x(t) = -2$ $\quad C)$ $\lim_{t\to\infty} x(t) = 4$

$D)$ $\lim_{t\to-\infty} x(t) = 4$ $\quad E)$ $\lim_{t\to-\infty} x(t) = \infty$ $\quad F)$ $\lim_{t\to\infty} x(t) = -\infty$

Factoring: $\frac{dx}{dt} = (x-2)(x-4)$. Observe that the DEQ has two equilibrium points $x = 2$ and $x = 4$. We want to check the domain on either side of these points. At our initial value $x = 1$, we note that the rate of change is positive $\frac{dx}{dt} = 3$. In between the equilibrium points $x = 3$, the function is decreasing $\frac{dx}{dt} = -1$, so $x = 2$ is stable. To the right of the larger equilibrium point at $x = 5$ the function is increasing $\frac{dx}{dt} = 3$, so the larger equilibrium point is unstable. So visually: $\rightarrow 2 \leftarrow 4 \rightarrow$

Therefore, as $t \to \infty$ from $x(0) = 1$, the function will increase until it gets to the equilibrium point $x(t) \to 2$. And, as $t \to -\infty$, the function will decrease. And since there are no more equilibrium points less than $x(0) = 1$, the function will decrease without bound, or $x(t) \to -\infty$. So F is the only option consistent with this.

Problem 2. A ski jumper launches from rest down a long ramp. Her acceleration is proportional to the difference between 40 m/s (her theoretical maximum speed due to air resistance and ramp slope) and her current velocity. After 10 seconds, her speed is measured to be 20 m/s. How long will it take for the jumper to reach 35 m/s?

$\frac{dv}{dt} = k(40-v) \quad \Rightarrow \quad \int \frac{1}{40-v} dv = k\int dt \quad \Rightarrow \quad -\ln|40-v| = kt + C_1 \quad \Rightarrow \quad 40 - v = C_2 e^{-kt}$

$\Rightarrow \quad v = 40 - C_2 e^{-kt}$. First init-cond: $0 = 40 - C_2 e^0 \quad \Rightarrow \quad C_2 = 40$.

2nd init-cond: $20 = 40 - 40e^{-10k} \quad \Rightarrow \quad e^{-10k} = \frac{1}{2} \quad \Rightarrow \quad -10k = \ln\frac{1}{2} \quad \Rightarrow \quad k = \frac{\ln 2}{10}$

Time till 35? $35 = 40 - 40e^{-\frac{\ln 2}{10}t} \quad \Rightarrow \quad \frac{5}{40} = e^{-\frac{\ln 2}{10}t} \quad \Rightarrow \quad \ln\left(\frac{5}{40}\right) = -\frac{\ln 2}{10}t \quad \Rightarrow \quad t = -\frac{10\ln\left(\frac{5}{40}\right)}{\ln 2} = 30.0$ secs.

Problem 3 **First solve** $f(x) = 0$ **to find the critical points of the autonomous DEQ** $\frac{dx}{dt} = f(x) = 7x - x^2 - 10$. **Then analyze the sign of** $f(x)$ **to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the DEQ. Next, solve the DEQ explicitly for** $x(t)$ **in terms of** t. **Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given DEQ, and verify visually the stability of each critical point**.

$7x - x^2 - 10 = (x-2)(x-5) = 0 \quad \Rightarrow \quad x \in \{2, 5\}$ are the critical points.

Then, looking at the sign of $f(x)$ on either sides of the critical points, we have:
$f(1) = -4 < 0, \;\; f(3) = 2 > 0, \;\; f(6) = -4 < 0$.

This gives us the phase diagram: $\leftarrow \; 2 \; \rightarrow \; 5 \; \leftarrow$

Stable critical point: $x = 5$.

Unstable critical point: $x = 2$.

Sink: Along the equilibrium solution $x(t) = 5$.

Source: Along the equilibrium solution $x(t) = 2$.

Solution: If $x_0 \notin \{2, 5\}$, then :

$\frac{dx}{dt} = 7x - x^2 - 10 \quad \Rightarrow \quad \frac{1}{(x-2)(x-5)} dx = -dt$

Partial fractions: $\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$ when $1 = A(x-5) + B(x-2) = (A+B)x - (5A+2B)$

$\Rightarrow \quad A + B = 0$ and $-5A - 2B = 1$

$\Rightarrow \quad B = -A, \quad -5A - 2(-A) = 1, \quad A = -\frac{1}{3}$ and $B = \frac{1}{3}$.

Therefore, $\frac{1}{(x-2)(x-5)} = \frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5}$

Continuing our integration: $\int \frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5} dx = -\int dt$ or $\int \frac{1}{x-5} - \frac{1}{x-2} dx = -3 \int dt$

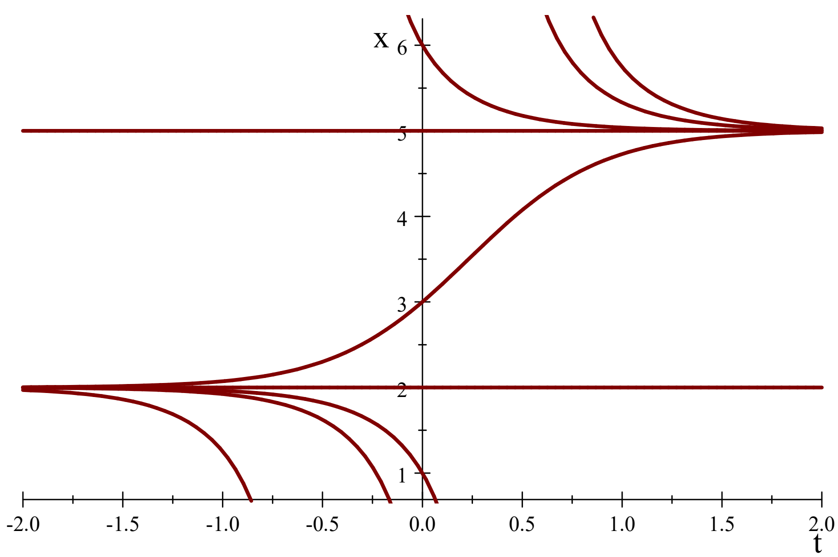
$\ln|x-5| - \ln|x-2| = -3t + C \quad \Rightarrow \quad \ln\left|\frac{x-5}{x-2}\right| = -3t + C$

$\frac{x-5}{x-2} = \pm e^C e^{-3t}$, and for any initial condition we have $\frac{x_0-5}{x_0-2} = \pm e^C$, so our solution becomes $\frac{x-5}{x-2} = \frac{x_0-5}{x_0-2} e^{-3t}$.

Solving explicitly for x: $\quad x - 5 = \frac{x_0-5}{x_0-2} e^{-3t}(x-2)$

$x - \frac{x_0-5}{x_0-2} e^{-3t} x = -2\frac{x_0-5}{x_0-2} e^{-3t} + 5$

$x = \frac{-2\frac{x_0-5}{x_0-2} e^{-3t} + 5}{\left(1 - \frac{x_0-5}{x_0-2} e^{-3t}\right)} = \frac{5(x_0-2) - 2(x_0-5)e^{-3t}}{(x_0-2) - (x_0-5)e^{-3t}} = \frac{5(x_0-2)e^{3t} - 2(x_0-5)}{(x_0-2)e^{3t} - (x_0-5)}$.



$\frac{5(x_0-2)e^{3t} - 2(x_0-5)}{(x_0-2)e^{3t} - (x_0-5)}$