2.1 Exercises - Solutions 💪

Problem 1 Solve the initial value problem: $\frac{dx}{dt} = 9 - 4x^2, \quad x(0) = 0.$
(Hint: you will need to use partial fractions).

$\int \frac{1}{(3+2x)(3-2x)}dx = \int 1dt$ (when $x \neq \pm\frac{3}{2}$).

But we would rather work with: $\int \frac{A}{3+2x} + \frac{B}{3-2x}dx = \int 1dt.$

So set: $A(3-2x) + B(3+2x) = 1.$ Collecting terms: $(3A+3B) + (2B-2A)x = 1.$

Comparing powers of x we get: $3A + 3B = 1$ and $2B - 2A = 0.$ So, $B = A,$ and $A = \frac{1}{6}.$

So now we can solve the integral: $\frac{1}{6}\int\left(\frac{1}{3+2x} + \frac{1}{3-2x}\right)dx = \int 1dt,$

$\Rightarrow \quad \int\left(\frac{1}{3+2x} + \frac{1}{3-2x}\right)dx = \int 6dt \quad \Rightarrow \quad \frac{1}{2}\ln(3+2x) - \frac{1}{2}\ln(3-2x) = 6t + C_0$

$\Rightarrow \quad \frac{1}{2}\ln\frac{3+2x}{3-2x} = 6t + C_0 \quad \Rightarrow \quad \ln\frac{3+2x}{3-2x} = 12t + C_1 \quad \Rightarrow \quad \frac{3+2x}{3-2x} = C_2e^{12t}.$

$x(0) = 0$ implies $1 = C_2,$ so $\frac{3+2x}{3-2x} = e^{12t}.$

To solve explicitly (not that you were asked to):
$3 + 2x = (3-2x)e^{12t} \quad \Rightarrow \quad 3 + 2x = 3e^{12t} - 2xe^{12t} \quad \Rightarrow \quad 2x + 2xe^{12t} = 3e^{12t} - 3$

$\Rightarrow \quad x(2 + 2e^{12t}) = 3(e^{12t} - 1) \quad \Rightarrow \quad x(t) = \frac{3\left(e^{12t}-1\right)}{2\left(e^{12t}+1\right)}.$

But how about, when $x = \pm\frac{3}{2}$? These possible solutions don't satisfy our initial condition $x(0) = 0.$

$x(t) = \frac{3\left(e^{12t}-1\right)}{2\left(e^{12t}+1\right)}$ OR $\frac{3+2x}{3-2x} = e^{12t}$

Problem 2 **Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$.**

Show that $P(t) = \frac{P_0}{1-kP_0t}$, **with** k **constant**: **Note that as** $t \to \frac{1}{kP_0}$ we have $P(t) \to +\infty$. **This is doomsday**.

$\beta := c_1P$ and $\delta := c_2P,$ so $\beta - \delta = (c_1 - c_2)P = kP.$

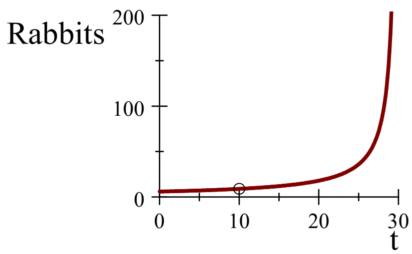
Using the DEQ $\frac{dP}{dt} = (\beta - \delta)P$ from the book, and substituting from above gives us: $P' = kP^2$ with k positive.

Using separation of variables: $\frac{1}{P^2}\frac{dP}{dt} = \frac{d}{dt}(-P^{-1}),$ so we have:

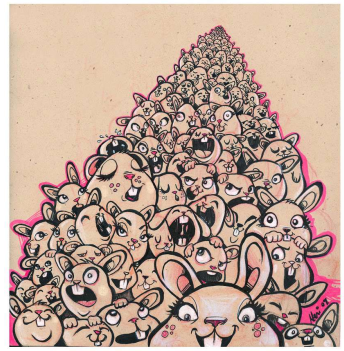
$\int \frac{d}{dt}(-P^{-1})dt = k\int dt \quad \Rightarrow \quad -P^{-1} = kt + C$

$\Rightarrow \quad P(t) = \frac{1}{C-kt}.$

The initial condition $P(0) = P_0$ then gives $P_0 = \frac{1}{C}$ or $C = \frac{1}{P_0}.$ So $P(t) = \frac{1}{\frac{1}{P_0}-kt} = \frac{P_0}{1-kP_0t}.$



Initial population: 6. $k = \frac{1}{180}$



doomsday of infinite bunnies

Problem 3 Logistic Equation, $\frac{dP}{dt} = kP(M - P)$.

Suppose that at time $t = 0$, half of a "**logistic**" **population** of 100,000 persons have heard a certain rumor, and that the number of those who have heard it is then increasing at the rate of 1,000 persons per day. How long will it take for this rumor to spread to 80% of the population?

Logistic means we have: $P' = kP(M - P)$. We'll work in thousands of persons.

What does P stand for? What is the carrying capacity?

The word problem indicates: $P(0) = 50$ and $P'(0) = 1$.

So $M = 100$ and $P' = kP(100 - P)$.

Substituting $P'(0) = 1$:

$1 = P'(0) = kP(0)(100 - P(0))$

$= 50k(100 - 50) = 2,500k$

$1 = 2,500k$ or $k = 0.0004$ and $P' = (0.0004)P(100 - P)$.

"**How long will it take for this rumor to spread to** 80% **of the population**?"

If t denotes the number of days until 80,000 people have heard the rumor, then Equation 7 in the text ($P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{-kMt}}$) gives...

$80 = \frac{100 \cdot 50}{50 + (100 - 50)e^{-0.04t}} = \frac{5000}{50 + 50e^{-0.04t}} \quad \Rightarrow \quad 50 + 50e^{-0.04t} = \frac{5000}{80} = 62.5$

$\Rightarrow e^{-0.04t} = \frac{12.5}{50} = \frac{1}{4}, \quad \Rightarrow \quad -0.04t = \ln\frac{1}{4} = -\ln 4$

$\Rightarrow t = \frac{\ln 4}{0.04} \approx 34.66$ days.

Thus the rumor will have spread to 80% of the population in a little less than 35 days.