1.6 Exercises - Solutions 💪

Problem 1 Find the general solution to: $xy' = y + 2\sqrt{xy}$

Solving for y', we find: $y' = \frac{y}{x} + \frac{2}{x}\sqrt{xy} = \frac{y}{x} + 2\sqrt{\frac{1}{x^2}}\sqrt{xy} = \frac{y}{x} + 2\sqrt{\frac{y}{x}}$, when $x \neq 0$.

In other words, it has scalar homogeneity, so we attempt the substitution: $v = \frac{y}{x}$, $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

So we have: $v + x\frac{dv}{dx} = v + 2\sqrt{v} \quad \Rightarrow \quad \int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$, when $v \neq 0$ or equivalently when $y(x) \neq 0$.

$\sqrt{v} = \ln x + C$, $\quad y = x(\ln x + C)^2$. Also note that the singular case $y(x) \equiv 0$ is also a solution.

Problem 2 Find the general solution to: $x^2y' = xy + y^2$.

Solving for y', we find: $y' = \frac{y}{x} + (\frac{y}{x})^2$, when $x \neq 0$.

In other words, it has scalar homogeneity, so we attempt the substitution: $v = \frac{y}{x}$, $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

So we have: $v + x\frac{dv}{dx} = v + v^2 \quad \Rightarrow \quad \int \frac{dv}{v^2} = \int \frac{1}{x}dx$, when $v \neq 0$ or equivalently when $y(x) \neq 0$.

$\Rightarrow -v^{-1} = \ln|x| + C$

$\Rightarrow \quad -\frac{x}{y} = \ln|x| + C \quad \Rightarrow \quad y = -\frac{x}{\ln|x| + C}$. Also note that the singular case $y(x) \equiv 0$ is also a solution.

Problem 3 Find the general solution to: $y^2y' + 2xy^3 = 6x$

Solving for y', we have: $y' = 6\frac{x}{y^2} - 2xy$, when $y(x) \neq 0$. We see this takes on the Bernoulli form.

Setting: $v := y^{1-n} = y^3$. We then multiply the given DEQ by 3, and we have: $v' + 6xv = 18x$.

Then we can calculate our integrating factor: $\rho = e^{\int 6xdx} = e^{3x^2}$.

We then have: $v = \frac{1}{\rho}\int (P(x)\rho)dx = 18e^{-3x^2}\int xe^{3x^2}dx$, $\quad u = 3x^2$, $\quad du = 6xdx$

$y^3 = 18e^{-3x^2}\int \frac{1}{6}e^u du = 3e^{-3x^2}(e^u + C) = 3e^{-3x^2}e^{3x^2} + 3Ce^{-3x^2}$

or $\quad y^3 = 3 + Ce^{-3x^2}$.

Problem 4 Verify $(4x - y)dx + (6y - x)dy = 0$ is exact; then solve it.

We verify: $\frac{\partial}{\partial y}(4x - y) = -1 = \frac{\partial}{\partial x}(6y - x)$, so the DEQ is exact.

$F = \int (4x - y)dx = 2x^2 - xy + g(y)$; $\quad F_y = -x + g'(y) = 6y - x = N$

$g'(y) = 6y$; $\quad g(y) = 3y^2$ (plus constant); $\quad$ So: $2x^2 - xy + 3y^2 = C$