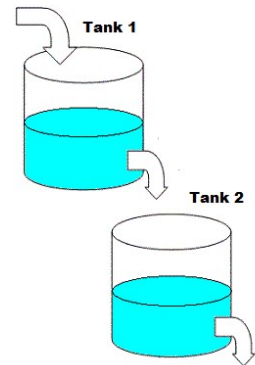


1.5 Exercises - Solutions

Problem 1

Consider the cascade of two tanks shown, with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) representing the volumes of solution (salt water) in the two tanks. Dissolved in the water of each tank is 50 lb of salt. The three flow rates indicated in the figure (w/curved arrows) are each 5 gal/min, with pure water flowing into tank 1.

a) Find the amount of salt in tank 1 at time t , represented as $x(t)$.



$$\frac{dx}{dt} = [IN] - [OUT] = (\text{incoming flow of liquid}) \cdot (\text{incoming concentration of salt}) - (\text{outgoing flow of liquid}) \cdot (\text{outgoing concentration of salt})$$

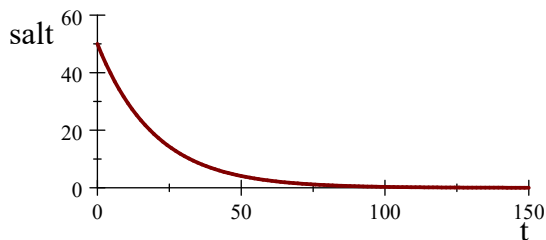
$$= 5 \cdot 0 - 5 \cdot \frac{x}{100} = -\frac{x}{20}. \quad (\text{incoming salt is zero since it is pure water})$$

$$\int \frac{d}{dt} \ln|x| dt = -\frac{1}{20} \int dt \quad \Rightarrow \quad \ln|x| = -\frac{t}{20} + c$$

$$\Rightarrow \quad x = Ce^{-\frac{t}{20}}, \text{ where } C > 0. \text{ (because a nonzero amount of salt is always positive)}$$

$$x(0) = 50, \text{ so } 50 = Ce^0 \text{ and } C = 50. \quad (\text{Initial conditions})$$

$$\text{Therefore, } x(t) = 50e^{-\frac{t}{20}}.$$



$$50e^{-\frac{t}{20}}$$

b) Suppose that $y(t)$ represents the amount of salt in tank 2 at time t . Solve for $y(t)$.

$$\frac{dy}{dt} = [IN] - [OUT]$$

$$= \frac{x}{20} - \frac{5y}{200} \quad (\text{we calculated [IN] in part a)}$$

$$= \frac{50e^{-\frac{t}{20}}}{20} - \frac{5y}{200} = \frac{5}{2}e^{-\frac{t}{20}} - \frac{y}{40}. \quad (\text{substitute in } x(t) \text{ from part a})$$

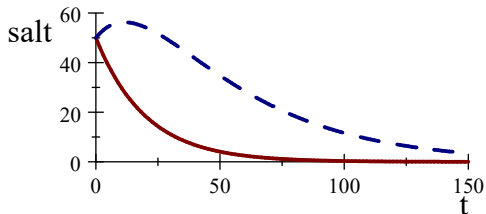
$$\frac{dy}{dt} + \frac{y}{40} = \frac{5}{2}e^{-\frac{t}{20}} \quad \Rightarrow \quad \rho = e^{\frac{1}{40} \int dt} = e^{\frac{t}{40}}$$

$$y = e^{-\frac{t}{40}} \frac{5}{2} \int e^{\frac{t}{40}} e^{-\frac{t}{20}} dt = \frac{5}{2} e^{-\frac{t}{40}} \int e^{-\frac{t}{40}} dt$$

$$y = \frac{5}{2} e^{-\frac{t}{40}} (-40) e^{-\frac{t}{40}} + e^{-\frac{t}{40}} C = -100e^{-\frac{t}{20}} + e^{-\frac{t}{40}} C.$$

Init-cond: $y(0) = 50 = -100e^0 + Ce^0$. So, $C = 150$.

And $y = -100e^{-\frac{t}{20}} + 150e^{-\frac{t}{40}}$.



$$50e^{-\frac{t}{20}} \text{ and } -100e^{-\frac{t}{20}} + 150e^{-\frac{t}{40}}$$

c) Finally, find the maximum amount of salt ever in tank 2.

$$y'(t) = \left(-\frac{1}{20}\right)(-100e^{-\frac{t}{20}}) + \left(-\frac{1}{40}\right)(150e^{-\frac{t}{40}}) = 5e^{-\frac{t}{20}} - \frac{15}{4}e^{-\frac{t}{40}} = 0$$

$$\Rightarrow 4e^{-\frac{2t}{40}} - 3e^{-\frac{t}{40}} = 0, \quad 4e^{-\frac{t}{40}} - 3 = 0 \quad (\text{multiplied through by } e^{\frac{t}{40}})$$

$$\Rightarrow e^{-\frac{t}{40}} = \frac{3}{4} \quad -\frac{t}{40} = \ln \frac{3}{4}$$

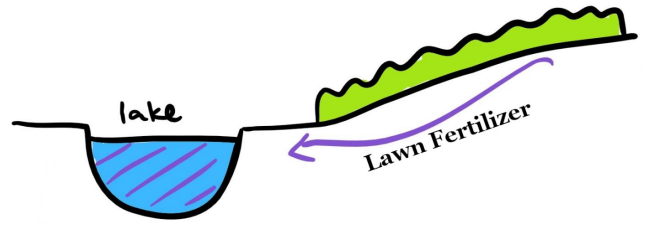
$$\Rightarrow t = -40 \ln \frac{3}{4} \approx 11.51 \text{ min.}$$

"Find the maximum amount of salt"

$$y_{\max} = y(11.51) = 56.25 \text{ lb.}$$

Note: Shortcut to explicit solution to linear 1st-order DEQ: $y = \frac{1}{\rho} \int \rho Q dx$.

Problem 2 Consider a lake with a volume of 8 billion cubic feet (ft^3) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million ft^3 of water with a pollutant concentration of 0.05% and an equal daily outflow of the (well-mixed) water in the lake. How long will it take to reduce the pollutant concentration in the reservoir to 0.10%?



Let $x(t)$ denote the amount of pollutants in the lake after t days.

We set up the differential equation in infinitesimal form by writing...

$$\frac{dx}{dt} = [IN] - [OUT].$$

$$IN = (\text{Concentration}) \times (\text{rate of input}) = (0.0005)(500). \quad (\text{counting in millions})$$

$$OUT = (\text{Concentration}) \times (\text{rate of output}) = \frac{x}{8000} \cdot 500. \quad (\text{The volume of the lake is } 8,000 \text{ } mft^3.)$$

$$\frac{dx}{dt} = [IN] - [OUT] = (0.0005)(500) - \frac{x}{8000} \cdot 500,$$

$$\text{which simplifies to } \frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}, \text{ or } \frac{dx}{dt} + \frac{1}{16}x = \frac{1}{4}.$$

First, calculate the integrating factor: $I(t) = e^{\int \frac{1}{16} dt} = e^{\frac{t}{16}}$.

Multiply both sides of the equation by the integrating factor

$$e^{\frac{t}{16}} \left(\frac{dx}{dt} + \frac{1}{16}x \right) = \frac{1}{4} e^{\frac{t}{16}}$$

Next, recognize the left-hand side is the derivative of the product, $xe^{\frac{t}{16}}$.

So, integrate the equation: $D_t(xe^{\frac{t}{16}}) = \frac{1}{4}e^{\frac{t}{16}}$, to get:

$$xe^{\frac{t}{16}} = \frac{1}{4} \int e^{\frac{t}{16}} dt = \frac{1}{4} (e^{\frac{t}{16}} \cdot 16) + C = 4e^{\frac{t}{16}} + C. \quad x(t) = 4 + Ce^{-\frac{t}{16}}.$$

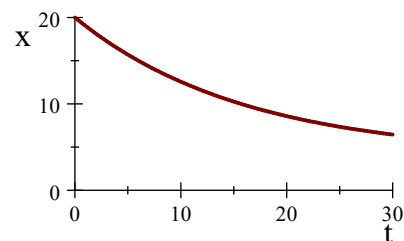
The initial amount $x(0)$ of pollutants is...

$$x_0 = (0.25\%)(8000) = (0.0025)(8000) = 20 \text{ } mft^3.$$

Plug-in our initial conditions: $x(0) = 4 + Ce^0 = 4 + C = 20$,

$$\Rightarrow C = 16.$$

$$x(t) = 4 + 16e^{-\frac{t}{16}}.$$



$$4 + 16e^{-\frac{t}{16}}$$

We want to know when $x(t) = (0.10\%)(8000) = (0.0010)(8000) = 8 \text{ mft}^3$.

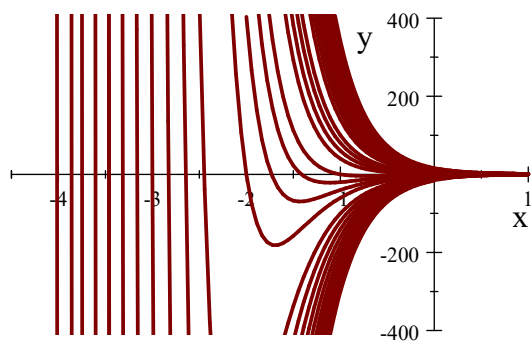
$$8 = 4 + 16e^{-\frac{t}{16}}, \quad \Rightarrow \quad \frac{1}{4} = e^{-\frac{t}{16}}, \quad \Rightarrow \quad e^{\frac{t}{16}} = 4$$
$$\Rightarrow \quad \ln 4 = \frac{t}{16}, \quad \Rightarrow \quad t = 16 \ln 4 \approx 22.2 \text{ days.}$$

Problem 3 Find the general solution of the differential equation: $y' = 2xe^{-3x} - 3y$.

$$y' + 3y = 2xe^{-3x}$$

$$\rho = e^{\int P(x)dx} = e^{\int 3dx} = e^{3x}. \quad (\text{observe that } P(x) = 3 \text{ NOT } 3y)$$

$$y = \frac{1}{\rho} \int \rho Q(x) dx = e^{-3x} \int (e^{3x} \cdot 2xe^{-3x}) dx$$
$$= e^{-3x} \int 2x dx = e^{-3x}(x^2 + c).$$



$e^{-3x}(x^2 + c)$ for various c