1.4 Exercises - Solutions

Problem 1 Find explicit particular solutions for the initial value problem:

$\frac{dy}{dx} = 2xy^2 + 3x^2y^2$, with init. cond. $y(1) = -1$.

$\frac{dy}{dx} = y^2(2x + 3x^2)$

$\frac{1}{y^2}\frac{dy}{dx} = 2x + 3x^2$, when $y(x) \neq 0$.

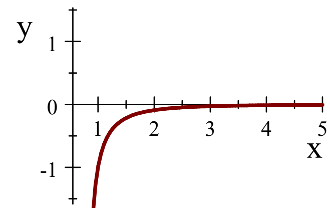
While $y(x) = 0$ does solve the DEQ, it does not fit the init-cond.

$\Rightarrow \quad \int \frac{d}{dx}(-\frac{1}{y})dx = \int(2x + 3x^2)dx$ (integrating factor $\frac{1}{y^2}$)

$\Rightarrow -\frac{1}{y} = x^2 + x^3 + C, \quad y(x) = -\frac{1}{x^2+x^3+C}$.

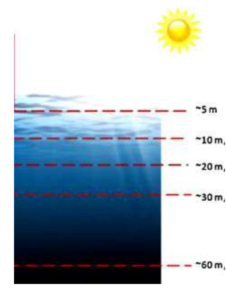
$-1 = -\frac{1}{1^2+1^3+C}, \quad 2 + C = 1, \quad C = -1.$ (applying the init-cond)

So, $y(x) = -\frac{1}{x^2+x^3-1}$.



$-\frac{1}{x^2+x^3-1}$

Underwater Light Intensity



Problem 2 **Natural (Exponential) Growth and Decay:** $\frac{dy}{dx} = ky$, where k is a constant.

The intensity I of light at a depth of x meters below the surface of a lake satisfies the differential equation: $\frac{dI}{dx} = -1.4I$.

Part *a*. At what depth is the light intensity I half of the surface light intensity $I_0 := I(0)$?

$\frac{1}{I}dI = -1.4dx$, for $I \neq 0$. (integrating factor $\frac{1}{I}$, for $I(x) \neq 0$, so not at night, which is a solution!)

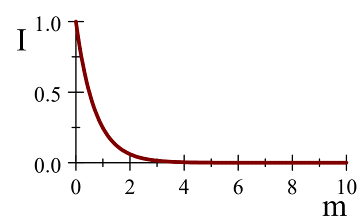
$\int \frac{d}{dx}\ln|I|dx = -1.4\int dx$

$\ln|I| = -1.4x + C$

$e^{\ln|I|} = e^C e^{-1.4x}$

The light intensity at a depth of x meters is therefore given by $I(x) = I_0 e^{-1.4x}$.

(how do I know $e^C = I_0$?)



$e^{-1.4x}$

Recall: "At what depth is the light intensity I, half of the surface light intensity, I_0 (where $x = 0$)?"

So, we solve the equation $\frac{1}{2}I_0 = I_0 e^{-1.4x}$.

$e^{-1.4x} = \frac{1}{2}$,

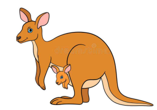
$-1.4x = \ln\frac{1}{2}, \quad \Rightarrow \quad x = \frac{-\ln 2}{-1.4} \approx 0.495\ m$ below the surface.

Part *b*. **What is the intensity at a depth of** $10\ m$ **(in terms of** I_0**)**? $\quad I(x) = I_0 e^{-1.4x}$

At depth 10 meters, the intensity is: $I(10) = I_0 e^{-1.4 \cdot 10} \approx (8.32 \times 10^{-7}) I_0$. (very dark)

Population Growth

Problem 3 For a population of kangaroos, assume that 10% of kangaroos become pregnant every year and have, on average, two joeys (babies) per litter. Also assume that 10% of kangaroos die annually.



a) Given an exponential growth model, **what is the DEQ which describes the population**?

Birth and death rates: $\beta = 0.2, \quad \delta = 0.1$.

So $k = \beta - \delta = 0.1$,

and $P'(t) = kP = 0.1P$.

b) **What is the general solution**?

Solving: $\frac{1}{P}\frac{dP}{dt} = 0.1$ (assuming $P(t) \equiv 0$, which is a solution!)

$\Rightarrow \quad \int \frac{d}{dt} \ln P dt = \int 0.1 dt$

$\Rightarrow \quad \ln P = 0.1t + c \Rightarrow P = Ce^{0.1t}$, where $C > 0$.

Note that since $P(t) \equiv 0$ is a singular solution, we can generate the general solution as $P = Ce^{0.1t}$, where $C \geq 0$.

c) If you know that in 2011 there were 34 million kangaroos, **how many kangaroos are predicted by this model this year**?

Let t represent the number of years since 2011. So we have the initial condition $P(0) = 34$.

$34 = Ce^0 \quad \Rightarrow \quad P = 34e^{0.1t}$

$P(10) = 34e^{0.1(15)} \approx 152.38$ million kangaroos!