1.3 Exercises - Solutions 💪

Existence and Uniqueness Theorem: Let: $\frac{dy}{dx} = f(x,y)$, with initial cond. $y(a) = b$.

Assume both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous on some rectangle R (containing (a,b) in its interior).

Then there exists a unique **local** solution $y(x)$ in some interval $x \in I$ (still containing a) within R, but possibly smaller than the width of R. Particularly, continuity of $f(x,y)$ guarantees existence on some I. And continuity of $\frac{\partial}{\partial y}f(x,y)$ guarantees uniqueness of that solution.

In addition to being continuous, if $\frac{\partial}{\partial y}f(x,y)$ is also **bounded** for all x *and* y, then **global** existence/uniqueness (on $\mathbb{R}$) of a solution y is guaranteed.

Problem 1 Determine whether existence of at least one solution of the initial value problem $y\frac{dy}{dx} = x - 1;\ y(1) = 0$ is guaranteed. If so, then is uniqueness of that solution also guaranteed?

$f(x,y) = \frac{x-1}{y}$, $\quad \frac{\partial}{\partial y}f = \frac{-(x-1)}{y^2}$ $\quad$ Continuous near $(1,0)$?

Neither $f(x,y) = \frac{x-1}{y}$ nor $\frac{\partial}{\partial y}f = \frac{-(x-1)}{y^2}$ is continuous near $(1,0)$, so the existence-uniqueness theorem guarantees nothing.

There still may be existence/uniqueness of a solution, but these tools don't tell us either way.

Problem 2 This problem will illustrate that if the hypotheses of the above **theorem** are NOT satisfied, then the initial value problem $y' = f(x,y)$, $y(a) = b$ may have either

- no solutions (no existence)
- finitely many solutions, or (existence, possibly uniqueness)
- infinitely many solutions. (no uniqueness)

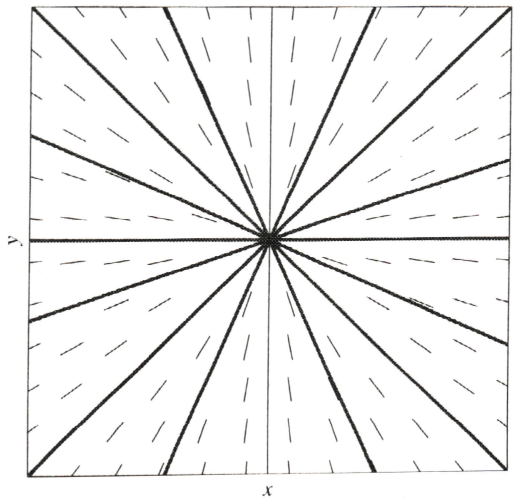
Verify that if k is a constant, then the function $y(x) = kx$ satisfies DEQ: $xy' = y$ for all x.

Taking the derivative of the solution, we have: $y' = (kx)' = k$.

Substituting this into our DEQ $xy' = x(kx)' = kx$, which is equal to $y = kx$.

Note that $y' = \frac{y}{x}$ is not continuous at $(0,b)$. So the theorem is NOT satisfied.

Construct a slope field with several of the straight-line solution curves.



Then determine how many different solutions the initial value problem: $xy' = y,\ y(a) = b$ **has for various** (a, b):

One, **none**, **or infinitely many**.

Note that $f(x,y) = \frac{y}{x}$ is certainly continuous when $x \neq 0$, as is $\frac{\partial}{\partial y} f = \frac{1}{x}$.

So our theorem guarantees us a unique solution when $a \neq 0$, but tells us nothing when $a = 0$.

However, we can visually verify above that there are infinitely many solutions when $a = 0, b = 0$.

When $a = 0, b \neq 0$, that $xy' = y$ has no solutions, because we end up with the false statement: $0y' = b \neq 0$.

So, the initial value problem has...

- a unique solution off the y-axis where $a \neq 0$;
- infinitely many solutions through the origin where $a = b = 0$;
- no solution if $a = 0$ but $b \neq 0$ (when (a, b) lies on the positive or negative y-axis).

Problem 3 Determine whether existence of at least one solution of the initial value problem $\frac{dy}{dx} = x^2 - y^2;\ y(0) = 1$ is guaranteed and, if so, whether uniqueness of that solution is guaranteed.

Both $f(x,y) = x^2 - y^2$ and $\frac{\partial f}{\partial y} = -2y$ are continuous near $(0, 1)$ (and everywhere since they are polynomials!), so the theorem guarantees both existence and uniqueness of a solution in some rectangle containing $x = 0$.

What about global existence?

$\frac{\partial f}{\partial y} = -2y$ is not bounded (goes to ∞ as $y \to \infty$), and $f(x,y)$ is not linear. So global existence is not guaranteed.