1.3 Exercises

Existence and Uniqueness Theorem: Let: $\frac{dy}{dx} = f(x,y)$, with initial cond. $y(a) = b$.

Assume both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous on some rectangle R (containing (a,b) in its interior).
Then there exists a unique **local** solution $y(x)$ in some interval $x \in I$ (still containing a) within R, but possibly smaller than the width of R. Particularly, continuity of $f(x,y)$ guarantees existence on some I.
And continuity of $\frac{\partial}{\partial y}f(x,y)$ guarantees uniqueness of that solution.

In addition to being continuous, if $\frac{\partial}{\partial y}f(x,y)$ is also **bounded** for all x *and* y, then **global** existence/uniqueness (on $\mathbb{R}$) of a solution y is guaranteed.

Problem 1 Determine whether existence of at least one solution of the initial value problem $yy' = x - 1;\ y(1) = 0$ is guaranteed. If so, then is uniqueness of that solution also guaranteed?

Problem 2 This problem will illustrate that if the hypotheses of the **Theorem** above are NOT satisfied, then the initial value problem $y' = f(x,y),\ y(a) = b$ may have either

- no solutions (no existence)
- finitely many solutions, or (existence, possibly uniqueness)
- infinitely many solutions. (no uniqueness)

a) **Verify that if k is a constant, then the function $y(x) = kx$ satisfies DEQ: $xy' = y$ for all x.**

b) **Construct a slope field with several of the straight-line solution curves**.

c) **Then determine how many different solutions the initial value problem**: $xy' = y,\ y(a) = b$ **has for various** (a, b): **One**, **none**, **or infinitely many**.

Problem 3 Determine whether existence of at least one solution of the initial value problem $\frac{dy}{dx} = x^2 - y^2;\ y(0) = 1$ is guaranteed and, if so, whether uniqueness of that solution is guaranteed.