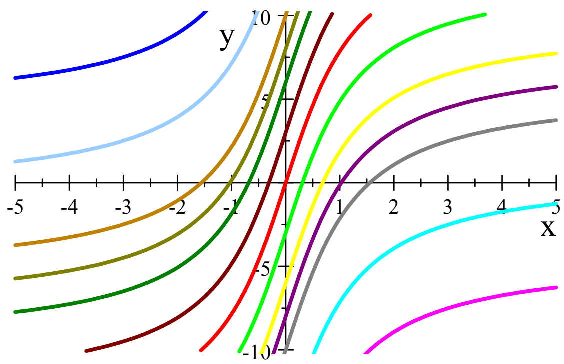
1.2 Exercises - Solutions 💪

Problem 1 Find the function $y = f(x)$ satisfying the DEQ $\frac{dy}{dx} = \frac{10}{x^2+1}$; with **initial condition** $y(0) = 0$.

$$y(x) = \int \frac{10}{x^2+1}\,dx$$

$$= 10\tan^{-1}x + C.$$ (If need be, review your inverse trigonometric derivatives)

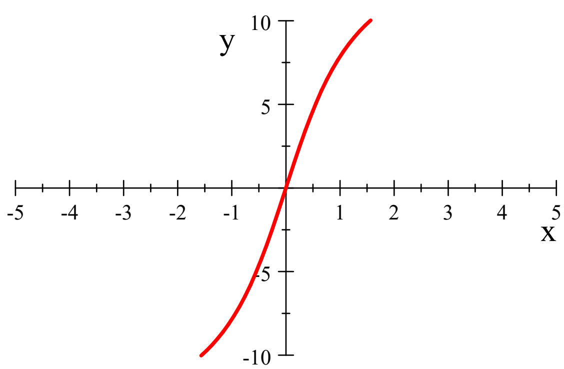


$10\tan^{-1}x + C,$ various values of C

Then, the substitution of init cond $x = 0,\ y = 0$ gives ...

$$0 = 10 \cdot 0 + C,$$

so $y(x) = 10\tan^{-1}x.$



$10\tan^{-1}x$

Problem 2

Find the position function $x(t)$ of a moving particle with the given acceleration: $a(t) = \frac{1}{(t+1)^3}$,

with **initial position** $x_0 = x(0) = 0$, and **initial velocity** $v_0 = v(0) = 0$.

$v(t) = \int (t+1)^{-3} dt$

Recall u-substitution: $u = t + 1 \quad \Rightarrow \quad du = dt \quad \Rightarrow \quad v(t) = \int u^{-3} du$

$= -\frac{1}{2}u^{-2} + C$

$= -\frac{1}{2}(t+1)^{-2} + C.$

Substituting init cond: $0 = -\frac{1}{2}(0+1)^{-2} + C = -\frac{1}{2} + C$. So, $C = \frac{1}{2}$.

This gives us: $v(t) = -\frac{1}{2}(t+1)^{-2} + \frac{1}{2}$.

And then: $x(t) = \int \left[-\frac{1}{2}(t+1)^{-2} + \frac{1}{2} \right] dt$

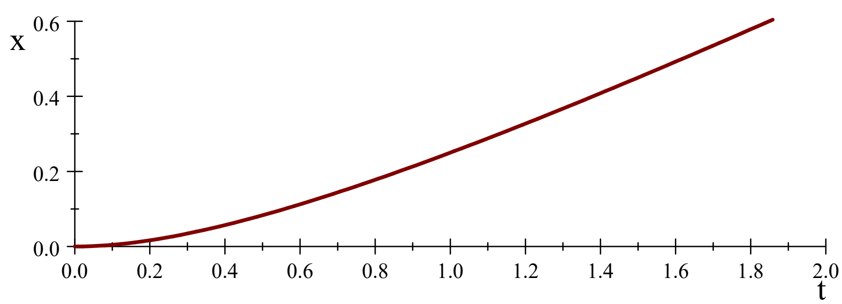
u –substitution: $u = t + 1 \quad \Rightarrow \quad du = dt \quad \Rightarrow \quad x(t) = \int \left[-\frac{1}{2}u^{-2} + \frac{1}{2} \right] du$

$= \frac{1}{2}u^{-1} + \frac{1}{2}u + C_1 = \frac{1}{2}(t+1)^{-1} + \frac{1}{2}(t+1) + C_1$ And then ...

Substituting init cond: $x(0) = 0$,

$0 = \frac{1}{2}(0+1)^{-1} + \frac{1}{2} \cdot (0+1) + C_1 = 1 + C_1$ and $C_1 = -1$.

So, $x(t) = \frac{1}{2}(t+1)^{-1} + \frac{1}{2}(t-1)$.



$\frac{1}{2}(t+1)^{-1} + \frac{1}{2}(t-1)$

Problem 3 Suppose a woman has enough "spring" in her legs to jump from the ground (on earth) to a height of 2.25 ft. Assume she jumps straight upward with the same initial velocity on the moon, where the surface gravitational acceleration is $5.3\ ft/s^2$. How high above the surface will she rise?

Eventually, we will need to solve an equation like: $x_m(t) = \int\int a_m(t)dtdt = Ct^2 + v_0 t + x_0$, where x_m, and a_m are position and velocity functions on the moon and where $x_0 = 0$ (the ground), and v_0 is the "same initial velocity" on both the Earth and the moon. So we need to find out what v_0 is.

First, let's figure things out on earth, where we know more. Like, we know: $a_e = g_e = -32\ ft/s^2$.
(gravitational acceleration on earth)

$$v_e(t) = -\int 32dt = -32t + v_0.$$

Now what? Well, we know something about when $x_e(t) = 2.25$ (her velocity is zero).
So, we better integrate again to get $x_e(t)$.

$$x_e(t) = \int v_e dt = \int(-32t + v_0)dt = -16t^2 + v_0 t + x_0 = -16t^2 + v_0 t.$$

Notice that if we now substitute $x_e(t) = 2.25$, we can solve for t in the previous equation, and substitute the result back into the velocity equation, where we know that the velocity is zero.

$$2.25 = -16t^2 + v_0 t \quad \Rightarrow \quad t = \frac{v_0 \pm \sqrt{v_0^2 - 64(2.25)}}{32} = \frac{v_0 \pm \sqrt{v_0^2 - 144}}{32}.$$

Plugging this back into our velocity equation, and setting the velocity is zero at the top of the jump…

$$0 = -32\left(\frac{v_0 \pm \sqrt{v_0^2 - 144}}{32}\right) + v_0 = -v_0 \pm \sqrt{v_0^2 - 144} + v_0 = \pm\sqrt{v_0^2 - 144}.$$

And so her initial velocity is: $v_0 = 12\ ft/\sec$.

Great, now back to the moon, we know: $a_m = -g_m = -5.3\ ft/s^2$.

So, $v_m(t) = -\int 5.3dt = -5.3t + v_0 = -5.3t + 12$.

Solving for t when $v_m = 0$ (top of the jump), we have $t = \frac{12}{5.3} = 2.2642$ sec.

Therefore, the height she reaches is: $x(t) = \int v_m dt = \int(-5.3t + 12)dt = -2.65t^2 + 12t + x_0$

$$= -2.65(2.2642)^2 + 12(2.2642) \approx 13.58\ ft.$$

(her height x_0 is zero since she's on the ground initially)