1.1 Exercises - Solutions

Problem 1 Verify by substitution that $y_1 = 0$ and $y_2 = \frac{\ln x}{x^2}$ are solutions to $x^2 y'' + 5xy' + 4y = 0$.

Looking at the first one, obviously: $y_1' = 0$ and $y_1'' = 0$.

Substituting them in: $x^2(0) + 5x(0) + 4(0) = 0.$ $\rightarrow \checkmark$

As for the second one: $y_2' = \frac{\frac{1}{x}x^2 - 2x\ln x}{x^4} = \frac{1-2\ln x}{x^3}$,

and $y_2'' = \frac{\left(-\frac{2}{x}\right)x^3 - (1-2\ln x)(3x^2)}{x^6} = \frac{-5+6\ln x}{x^4}$.

So, $x^2\left(\frac{-5+6\ln x}{x^4}\right) + 5x\left(\frac{1-2\ln x}{x^3}\right) + 4\left(\frac{\ln x}{x^2}\right) = \frac{-5+6\ln x}{x^2} + \frac{5-10\ln x}{x^2} + \frac{4\ln x}{x^2} = 0.$ $\rightarrow \checkmark$

Problem 2 Verify that $y(x) = x^3(C + \ln x)$ satisfies $xy' - 3y = x^3$.

Then, determine a value of the constant C so that $y(x)$ satisfies the initial condition $y(1) = 17$.

Taking a derivative: $y' = 3x^2(C + \ln x) + x^3\left(\frac{1}{x}\right) = 3x^2(C + \ln x) + x^2$.

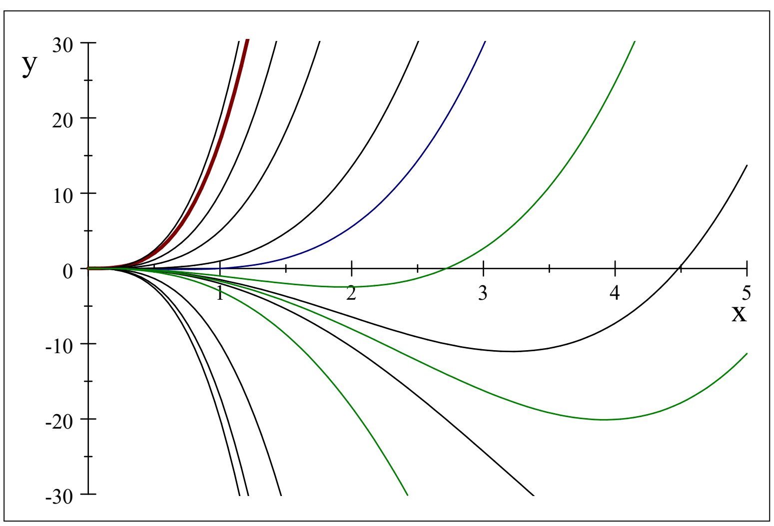
Substituting it in: $xy' - 3y = x[3x^2(C + \ln x) + x^2] - 3[x^3(C + \ln x)]$

$= 3x^3(C + \ln x) + x^3 - 3x^3(C + \ln x) = x^3.$ $\checkmark$

And, $y(x) = x^3(C + \ln x)$, with initial condition $y(1) = 17$ is:

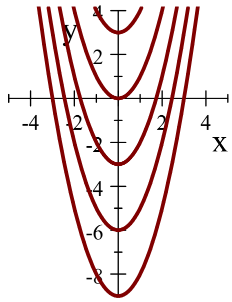
$17 = 1^3(C + \ln 1) = C.$ So, $C = 17$.

Graphing the **family** of $x^3(C + \ln x)$ along with **particular** solution (in red) $y(x) = x^3(17 + \ln x)$:

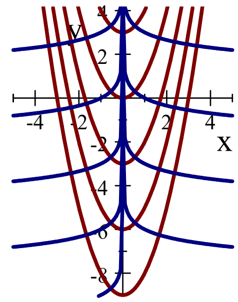


$y(x) = x^3(C + \ln x)$

Problem 3 When a graph of $g(x)$ intersects the graphs of functions with the form $y = x^2 + k$ (k is any constant), it does so perpendicularly (the two graphs are normal to each other). Write a differential equation of the form $g' = f(x)$ having the function g as its solution (or as one of its solutions).



$x^2 + k$



$x^2 + k$ and $g(x)$

We want the slopes of the two functions to be negative reciprocals.

$y' = D_x(x^2 + k) = 2x$

Negative Reciprocal: $g' = -\frac{1}{2x}$

Problem 4 Write a differential equation that is a mathematical model of the following situation.

There is a city which has a fixed population of P people.

N is the number of persons who have heard a certain rumor.

The time rate-of-change of N, is proportional to the number of those who have not heard the rumor.

$\frac{dN}{dt} = k(P - N)$ where k is some constant.