Differential Equations and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Laplace Transform of a Derivative
- Laplace Transform of a System
- Additional Laplace Transform Techniques
- Transforms of Integral Functions

11.1: Power Series Method

We've learned how to solve homogeneous linear DEQs with **constant coefficients** (characteristic equation or Laplace), but what if the coefficients are not constant?

We *did* solve DEQs w/variable coefficients with the **variation of parameters** method. However, that method has some limitations:

the need to find the homogeneous soln, the ability to integrate potentially complicated expressions, etc.

In this section we will find a more flexible method.

DEQs with **variable coefficients** come up in quantum mechanics, astrophysics, acoustics/optics, fluid dynamics, biology/medicine.

For most of these, we must use the the **power series method**.

Review

Recall a **power series** centerd at $x = a$ has the form: $\Sigma_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \ldots + c_n(x-a)^n + \ldots$.

Special case, when $a = 0$, we have: $\Sigma_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \ldots + c_n x^n + \ldots$.

What we wrote above is quite symbolic, but it certainly *looks* like a function $f(x)$ of some kind.

Howver, we run into the danger of the function returning infinity when we set x equal to certain numbers.

For the values of x such that the sum is finite, we say the series **converges**.

Similarly, we say the series **converges** on an interval I, provided the series is finite for **every** x in that interval.

And by converges, we mean: $\Sigma_{n=0}^{\infty} c_n x^n = \lim_{N\to\infty} \Sigma_{n=0}^{N} c_n x$ is finite (aka: defined).

On that interval, we consider the sum to be a function: $f(x) = \sum_{n=0}^{\infty} c_n x^n$, and refer to it as the *power series representation* of $f(x)$ on I. There are many commonly used functions which have a power series representation.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ (the sum converges to e^x for all x as $n \to \infty$)

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots$ (converges for all x)

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ (converges for all x)

$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ (converges if $|x| < 1$, diverges if $|x| > 1$)

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ (converges if $|x| < 1$, diverges if $|x| > 1$)

Recall: $0! = 1$ and $x^0 = 1$ for all x, including $x = 0$.

Also recall we can generalize these with the change variable: $x \to f(x)$, for example:

$$e^{7x} = \sum_{n=0}^{\infty} \frac{(7x)^n}{n!} = 1 + 7x + \frac{(7x)^2}{2!} + \frac{(7x)^3}{3!} + \ldots$$

How do we know that the above summations are equal to their associated functions? The sums are often *derived* as a Taylor series.

Recall: A **Taylor series** with **center** $x = a$ of the function f is the power series:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Of course, this assumes that f is infinitely differential at $x = a$.

For $x = 0$, this becomes: $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$.

Why choose one center over another?
The series converges more quickly to the true value of the function near the center, than farther away from it.

So, in practical applications where you can only calculate a finite number of terms in the Taylor series, you'll want to center your Taylor series in the middle of the region you're most interested in.

Example: What if we wish to find the Taylor series for $f(x) = e^x$ at $x = 0$.

Observe that $f^{(n)}(x) = e^x$ for all n, thus $f^{(n)}(0) = 1$. Therefore, the above Taylor series reduces to the power series displayed above for e^x.

Power Series Operations

Definition (**analytic function**): If the Taylor series of a function $f(x)$ converges to the original function $f(x)$ for all x in some open interval containing a, we say $f(x)$ is *analytic* at $x = a$.

Example: all polynomials, all rational functions (wherever the denominator is not zero) are analytic.

Also, if f and g are both analytic at $x = a$, then so is their sum $f + g$, their product $f \cdot g$, and their quotient $\frac{f}{g}$ (if $g(a) \neq 0$).

How does this come in handy? There are some functions whose series are very difficult to calculate in the straightforward manner (like $\tan x$). Instead, we can calculate the Taylor series for $\sin x$ and $\cos x$, and simply divide them.

But this does beg the question, how do you do arithmetic with power series?

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = (a_0 + a_1 x + \dots)(b_0 + b_1 x + \dots)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \dots$$

These both converge on any open interval in which both of the original series converged.

Example: $\sin x \cos x = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right)\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right)$

$$= x + \left(-\frac{1}{3!} - \frac{1}{2!}\right)x^3 + \left(\frac{1}{5!} + \frac{1}{4!} + \frac{1}{3!2!}\right)x^5 + \dots$$

$$= x - \frac{4}{3!}x^3 + \frac{16}{5!}x^5 - \dots$$

$$= \frac{1}{2}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots\right] = \frac{1}{2}\sin 2x, \text{ for all } x.$$ (we just derived a trig identity!!)

Power Series Method

Great! But how can we use all of this to solve DEQs?

Given a DEQ, the power series method is to assume the solution y has the form of a power series: $y = \sum_{n=0}^{\infty} c_n x^n$. Here, we are assuming y is analytic on some interval I of interest.

So, we take the appropriate number of derivatives of this power series, substitute them into our DEQ, and try to determine what the coefficients c_n must be to satisfy the DEQ.

But wait! How do you take derivative of a series?!?

new tool

Theorem (**Termwise Differentiation of Power Series**): If the power series representation $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$ converges on the open interval I, then f is differentiable on I and:
$f'(x) = \sum_{n=0}^{\infty} c_n (x^n)' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots$ at each point of I.

Example: Recall $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$, when $|x| < 1$.

And note (due to the quotient or power rule) that $\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$. But what is the series representation of this expression?

Using the above thm, we have: $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \ldots$, when $|x| < 1$. □

Before we start applying this knowledge to solve DEQs, we need a couple more thms.

new tool

The first one tells us that the Taylor series presented above is the *only* power series that represents the function f.
This will be useful for identifying the coefficients a_n in $\sum_{n=0}^{\infty} a_n x^n$.

Theorem (**Identity Principle**): If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for every point in some interval I, then $a_n = b_n$ for all n.

In particular, if $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x in some I, then $a_n = 0$, for all n.

Example: Solve $y' + 2y = 0$ using the power series method.

We substitute: $y = \sum_{n=0}^{\infty} a_n x^n$ and $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ into the DEQ, obtaining

$$\sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0. \qquad (*)$$

Recall from prior math, that when you're trying to solve a polynomial equation for the coefficients,
you do so by comparing powers of x.

In this situation, however, notice that x is presented with different exponents, depending on the series: x^{n-1} vs x^n.

We can rectify this by "**shifting the index**" of one of the series.
Merely for aesthetic reasons, I will attempt to shift the x^{n-1} summation to match the x^n one.

new tool

How do you do this? With a process similar to change of variables.
In the x^{n-1} series, everywhere I see an n (even in the bounds of summation), I will replace it with an $n+1$.
Doing this, gives: $\sum_{(n+1)=1}^{\infty} (n+1) a_{n+1} x^{(n+1)-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$.

Now we have a series involving x^n.
However, as a final check you should make sure that this new series produces the same sum as the original series.

Note that:

$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$, and
$\sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$ ✓

Looking again at $(*)$, we now have:

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} 2a_n x^n = \sum_{n=0}^{\infty}((n+1)a_{n+1} + 2a_n)x^n = 0.$$

(because the bounds of summation were the same, we could combine them)

If this equation is to hold, we need every power of x to equal zero, or $a_{n+1}(n+1) + 2a_n = 0$ for all $n \geq 0$.

Therefore: $a_{n+1} = -\frac{2a_n}{n+1}$.

This equation is referred to as a **recurrence relation**.

Note, if we happen to know a_0 we can use this to obtain a_1, then a_2, and so on.

Indeed, observe: $a_1 = -\frac{2a_0}{1}$,

$$a_2 = -\frac{2a_1}{2} = -\frac{2}{2}\left(-\frac{2a_0}{1}\right) = \frac{2^2 a_0}{1 \cdot 2},$$

$$a_3 = -\frac{2a_2}{3} = -\frac{2}{3}\left(\frac{2^2 a_0}{1 \cdot 2}\right) = -\frac{2^3 a_0}{3!},$$

Now we see the pattern: $a_n = (-1)^n \frac{2^n a_0}{n!}$, $n \geq 0$.

So, our soln becomes: $y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(-1)^n \frac{2^n a_0}{n!} x^n$.

$$= a_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = a_0 e^{-2x}. \qquad \text{(became power series for } e^{-2x}\text{)}$$

The soln converges, since we saw above that the power series for the exponential function converges everywhere.

Note that above we shifted the indexing by one. You can similarly shift the index of a series by any integer k by doing a similar change $n \to n + k$.

We lucked out with this example, because we recognized at the end of the process that the resulting series solution was the exponential function.

However, more frequently you are left with a series solution whose convergence is unknown.
So how do you determine if the solution you found is convergent?

Recall: **Theorem** (**Radius of Convergence** ρ): Given $\Sigma c_n x^n$, suppose that $\rho = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$ exists (ρ is finite) or is infinite (we write $\rho = \infty$). Then:

- If $\rho = 0$, the series diverges for all $x \neq 0$.
- If $0 < \rho < \infty$, then $\Sigma c_n x^n$ converges if $|x| < \rho$, and diverges if $|x| > \rho$.
- If $\rho = \infty$, the series converges for all x.

Definition (**Radius of Convergence**): ρ is called the radius of convergence of the power series $\Sigma c_n x^n$.

Back to our Example: $\rho = \lim_{n\to\infty} \left| \dfrac{\frac{(-1)^n 2^n c_0}{n!}}{\frac{(-1)^{(n+1)} 2^{(n+1)} c_0}{(n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2} \right| = \infty.$

So, as we had already concluded, the series converges for all x.

Example: Solve $x^2 y' = y - x - 1$ using the power series method.

We substitute: $y = \sum_{n=0}^{\infty} a_n x^n$ and $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ into the DEQ, obtaining

$$x^2 \sum_{n=1}^{\infty} a_n n x^{n-1} = -x - 1 + \sum_{n=0}^{\infty} a_n x^n. \qquad (*)$$

$$\sum_{n=1}^{\infty} a_n n x^{n+1} = -x - 1 + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$ (LHS: absorbed the x^2. RHS: peeled off the $n = 0, 1$ terms)

Now, both series start at x^2, so just looking at the constant and x term, we have:

$a_0 = 1, \; a_1 = 1.$

Substituting $n \to n-1$ on the LHS, the equation becomes: $\sum_{n=2}^{\infty} a_{n-1}(n-1)x^n = -x - 1 + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$

Now comparing the x^n term, we have: $a_n = a_{n-1}(n-1)$ for $n \geq 2$.

Looking for a pattern, we find: $a_2 = 1 \cdot a_1 = 1, \quad a_3 = 2 \cdot a_2 = 2, \quad a_4 = 3 \cdot a_3 = 6.$

Notice this is: $a_2 = 1!, \quad a_3 = 2!, \quad a_4 = 3!.$

So we find the pattern: $a_n = (n-1)!$ for $n \geq 2$.

Thus: $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 1 + x + \sum_{n=2}^{\infty} (n-1)! x^n.$

Radius of convergence?

$$\rho = \lim_{n\to\infty} \frac{(n-1)!}{n!} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

So the series converges only for $x = 0$. (!?!)

This means that the DEQ does not have a convergent power series solution of the form we assumed: $y = \Sigma c_n x^n$.
When we write this series solution, we are making an assumption which might be false. □

Exercises 11.1

What did we learn?

- Power Series Operations/Method
- Termwise Differentiation of Power Series
- Recurrence Relations
- Radius of Convergence