10.3: Translation and Partial Fractions

To solve a linear differential equation with constant coefficients, the easiest method is often to use a Laplace transform. However, this often means applying the inverse Laplace transform to rational functions: $\frac{P(s)}{Q(s)}$.

The previous sections showed us how to apply inverse transforms to a limited number of rational functions. Therefore, we need to expand our abilities (with advanced partial fractions) to alter arbitrary rational functions into one of the forms we recognize from the table ($\frac{n!}{s^{n+1}}$, $\frac{s}{s^2+k^2}$, $\frac{1}{s-a}$, etc.).

Partial Fractions when denominator has factor $(s-a)$ **with multiplicity** n :
$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \ldots + \frac{A_n}{(s-a)^n}$ where $A_1, \ldots, A_n$ are constants.

Partial Fractions when denominator has irreducible quadratic factor $(s-a)^2+b^2$ **with multiplicity** n :
$\frac{A_1 s+B_1}{(s-a)^2+b^2} + \frac{A_2 s+B_2}{\left[(s-a)^2+b^2\right]^2} + \ldots + \frac{A_n s+B_n}{\left[(s-a)^2+b^2\right]^n}$ where $A_1, B_1, \ldots, A_n, B_n$ are constants.

So we must first break up rational functions ($\frac{P(s)}{Q(s)}$) using partial fractions ($\frac{P(s)}{Q(s)} = \frac{P_1(s)}{Q_1(s)} + \frac{P_2(s)}{Q_2(s)} + \ldots + \frac{P_n(s)}{Q_n(s)}$), and then apply inverse Laplace transforms to the individual terms. To accomplish the second part, we need the following…

Theorem: Translation on the s-Axis.

If $F(s) = L\{f(t)\}$ exists for $s > c$, then $L\{e^{at}f(t)\}$ exists for $s > a+c$, and $L\{e^{at}f(t)\} = F(s-a)$. Equivalently, $L^{-1}\{F(s-a)\} = e^{at}f(t)$.

Therefore,

$f(t)$	$F(s) = L\{f(t)\}$	
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	$s > a$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$	$s > a$

Problem: **#8** Apply the translation theorem to find the inverse Laplace transform of: $F(s) = \frac{s+2}{s^2+4s+5}$.

Completing the Square: (s^2+4s+5): $\frac{b}{2} = \frac{4}{2} = 2$, so…

$F(s) = \frac{s+2}{(s+2)^2+1}$, so... $L^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) =$?

$= e^{-2t}\cos t.$

Problem: **#26** Use the factorization $s^4+4a^4 = (s^2-2as+2a^2)(s^2+2as+2a^2)$ to derive the inverse Laplace transform: $L^{-1}\left\{\frac{1}{s^4+4a^4}\right\} = \frac{1}{8a^3}[e^{at}\sin at + e^{-at}\sin at + e^{-at}\cos at - e^{at}\cos at]$.

$\frac{1}{s^4+4a^4} = \frac{1}{(s^2-2as+2a^2)(s^2+2as+2a^2)}$

$= \frac{A+Bs}{s^2-2as+2a^2} + \frac{C+Ds}{s^2+2as+2a^2}$

$1 = (A+Bs)(s^2+2as+2a^2)+(C+Ds)(s^2-2as+2a^2)$

$1 = (As^2+2aAs+2a^2A)+(Bs^3+2aBs^2+2a^2Bs)+(Cs^2-2aCs+2a^2C)+(Ds^3-2aDs^2+2a^2Ds)$

$1 = (B+D)s^3+(A+2aB+C-2aD)s^2+(2aA+2a^2B-2aC+2a^2D)s+(2a^2A+2a^2C)$

$B+D=0,\quad A+2aB+C-2aD=0,\quad 2aA+2a^2B-2aC+2a^2D=0,\quad 2a^2A+2a^2C=1$

$A = \frac{1}{2a^2}-C,\qquad \left(\frac{1}{2a^2}-C\right)+2aB+C-2a(-B) = \frac{1}{2a^2}+4aB = 0,\ B = -\frac{1}{8a^3}$

$2a\left(\frac{1}{2a^2}-C\right)+2a^2\left(-\frac{1}{8a^3}\right)-2aC+2a^2\left(\frac{1}{8a^3}\right) = \frac{1}{a}-4aC = 0$

$C = \frac{1}{4a^2},\ A = \frac{1}{2a^2}-\frac{1}{4a^2} = \frac{1}{4a^2}.$

So, $A = \frac{1}{4a^2},\ B = -\frac{1}{8a^3},\ C = \frac{1}{4a^2},\ D = \frac{1}{8a^3}.$

Therefore, $\frac{1}{s^4+4a^4} = \frac{A+Bs}{s^2-2as+2a^2}+\frac{C+Ds}{s^2+2as+2a^2} = \frac{\left(\frac{1}{4a^2}\right)+\left(-\frac{1}{8a^3}\right)s}{s^2-2as+2a^2}+\frac{\left(\frac{1}{4a^2}\right)+\left(\frac{1}{8a^3}\right)s}{s^2+2as+2a^2}$

$= \frac{1}{8a^3}\left(\frac{2a-s}{s^2-2as+2a^2}+\frac{2a+s}{s^2+2as+2a^2}\right)$

Completing the Square: $s^2 \pm 2as+2a^2 = (s\pm a)^2+a^2$

$= \frac{1}{8a^3}\left(\frac{2a-s}{(s-a)^2+a^2}+\frac{2a+s}{(s+a)^2+a^2}\right) = \frac{1}{8a^3}\left(-\frac{s-a-a}{(s-a)^2+a^2}+\frac{s+a+a}{(s+a)^2+a^2}\right)$

$= \frac{1}{8a^3}\left(\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)^2+a^2}+\frac{s+a}{(s+a)^2+a^2}-\frac{s-a}{(s-a)^2+a^2}\right)$

So, $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{8a^3}\left[L^{-1}\left(\frac{a}{(s-a)^2+a^2}\right)+L^{-1}\left(\frac{a}{(s+a)^2+a^2}\right)+L^{-1}\left(\frac{s+a}{(s+a)^2+a^2}\right)-L^{-1}\left(\frac{s-a}{(s-a)^2+a^2}\right)\right]$

$= \frac{1}{8a^3}\left[e^{at}L^{-1}\left(\frac{a}{s^2+a^2}\right)+e^{-at}L^{-1}\left(\frac{a}{s^2+a^2}\right)+e^{-at}L^{-1}\left(\frac{s}{s^2+a^2}\right)-e^{at}L^{-1}\left(\frac{s}{s^2+a^2}\right)\right]$

$= \frac{1}{8a^3}[e^{at}\sin at+e^{-at}\sin at+e^{-at}\cos at-e^{at}\cos at].$

Problem: **#34** Use Laplace transforms to solve the initial value problem…

$x^{(4)}+13x''+36x = 0,\qquad x(0) = x''(0) = 0,\qquad x'(0) = 2,\qquad x'''(0) = -13.$

$L^{-1}\{x''\} = ?$

$L^{-1}\{x''\} = s^2X(s)-sx(0)-x'(0) = s^2X(s)-2$

$L^{-1}\{x^{(4)}\} = ?$

$$L^{-1}\{x^{(4)}\} = s^4X(s) - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0) = s^4X(s) - 2s^2 + 13$$

$$[s^4X(s) - 2s^2 + 13] + 13[s^2X(s) - 2] + 36X(s) = 0$$

$$(s^4 + 13s^2 + 36)X(s) - (2s^2 + 13) = 0$$

$$X(s) = \frac{2s^2+13}{s^4+13s^2+36} = ?$$

$$= \frac{2s^2+13}{(s^2+4)(s^2+9)} = ?$$

$$= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\begin{aligned} 2s^2 + 13 &= (As + B)(s^2 + 9) + (Cs + D)(s^2 + 4) \\ &= As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D \\ &= (A + C)s^3 + (B + D)s^2 + (9A + 4C)s + (9B + 4D) \end{aligned}$$

So, $A + C = 0, \quad B + D = 2, \quad 9A + 4C = 0,$ and $9B + 4D = 13.$

$A = -C, \quad -9C + 4C = 0, \quad C = 0, \quad A = 0, \quad B = 2 - D, \quad 9(2 - D) + 4D = 13,$

$18 - 9D + 4D = 13, \quad D = 1, \quad B = 1.$

$$X(s) = \frac{1}{s^2+4} + \frac{1}{s^2+9}$$

$$x(t) = L^{-1}\left(\frac{1}{s^2+4}\right) + L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t.$$