Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Almost Linear Systems
- Isolated Critical Pt, Jacobian matrix
- Stability of Almost Linear Systems

10.1: Laplace Transform Methods

We know DEQs $x^{(n)} = f(x^{(n-1)}, \ldots, x'', x', x, t)$ can be difficult/imposible to solve.
But what if we could transform a DEQ into an algebra problem?

In this section, we learn of a transformation which does just this. We then apply it in §10.2 to DEQs.
This method is particularly useful when the non-homogeneous term has discontinuities, which occurs in mass–spring and electric circuit systems (among others).

Given any function $f(t)$ on $t \geq 0$, let's define a "**Laplace transform**" as:
$F(s) := \mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} f(t)\,dt,$ for all s where the improper integral converges.

Recall that in order to evaluate an improper integral $\int_0^\infty g(t)\,dt$, we evaluate the limit $\lim_{b\to\infty} \int_0^b g(t)\,dt$.

If this limit exists, we say the integral converges.

Observe we started with an expression $f(t)$ in t, but end up with an expression $F(s)$ in s.

Example: What is the Laplace transform of $f(t) = e^{at}$ for $t \geq 0$? $F(s) = \mathcal{L}\{e^{at}\} =$

$$\int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_{t=0}^{\infty}$$

If $s - a > 0$, then $e^{-(s-a)t} \to 0$ as $t \to \infty$. (otherwise, it doesn't converge)

So it follows that $F(s) = \mathcal{L}\{e^{at}\} = -\frac{0}{s-a} - \left(-\frac{1}{s-a}\right) = \frac{1}{s-a}$ for $s > a$.

Don't let the fact that this is limited to functions starting at $t = 0$ trouble you, in practical applications we are usually looking at some phenomenon that has a starting time. We can always shift our function to make this time be zero.
There are also other techniques to get around this seeming limitation.

Once we learn a few tools and properties, we will be able to apply the transform to each term in a DEQ, and then solve the resulting algebraic problem.

Tools and Properties

A useful tool we'll need is a generalization of the factorial: $G(n) := n!$ The factorial's domain is discrete on the positive integers, but what if we could define a "*gamma function*" $\Gamma(x)$, whose domain involves real x when $x > 0$.

Gamma Function: $\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}dt$, from which you get (give it a try) $\Gamma(1) = 1, \Gamma(2) = 1, \Gamma(3) = 2, \Gamma(4) = 6$, etc.

Note this is reminiscent of the factorial, but shifted by one:
$G(0) = 0! = 1,\ G(1) = 1! = 1,\ G(2) = 2! = 1 \cdot 2,\ G(3) = 3! = 3 \cdot 2 \cdot 1,\ \ldots$ For more on $\Gamma(x)$ see §11.4.

So, for $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$.

In particular $\Gamma(n+2) = (n+1)! = (n+1) \cdot n! = (n+1)\Gamma(n+1)$
$= (n+1)n \cdot (n-1)! = n(n+1)\Gamma(n)$.

For example: $\Gamma(5) = 4 \cdot \Gamma(4) = 4 \cdot 3 \cdot \Gamma(3) = \ldots = 4!$, and $\Gamma(1) = 0!$, etc.

However, unlike the factorial function, Γ is continuous, so we can also evaluate $x \in \mathbb{R}$ when $x > 0$.
Particularly for fractions: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, so $\Gamma\left(\frac{5}{2}\right) =$??

$$= \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2} \cdot \tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \tfrac{3}{4}\sqrt{\pi}.$$

We will want to apply the Laplace transform to power functions t^a,
which turn out to be most conveniently expressed in terms of these gamma functions.

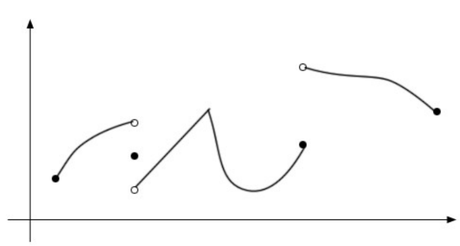
Linearity of Laplace Transforms: $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$. (this follows since integrals are linear)

Inverse Laplace Transforms: Given $F(s) = \mathcal{L}\{f(t)\}$, then we call $f(t)$ the **inverse Laplace transform** $\mathcal{L}^{-1}$ of $F(s)$, so: $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Linearity of Inverse Transforms: $\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$.
(follows since the inverse of a linear operator is also linear)

Piecewise Continuous Functions

Usefully, we can even apply Laplace transforms to some discontinuous functions!



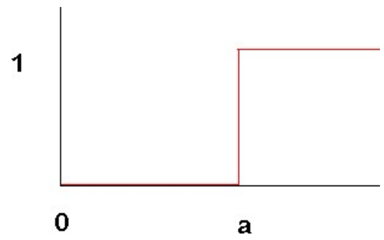
piecewise continuous function

$f(t)$ is **piecewise continuous** on $[a, b]$ if you can carve up $[a, b]$ into a **finite** number of sub-intervals such that:

- $f(t)$ is continuous on the interior of each of these subintervals.
- $f(t)$ has a finite limit as t approaches each end point of each subinterval.

Furthermore, we say that $f(t)$ is piecewise continuous on $[0,\infty)$ if it is piecewise continuous on every $[0,b]$ where $b > 0$.

Example: A **Unit Step Function** is $u_a(t) := \begin{cases} 0 & \text{for } t < a, \\ 1 & \text{for } t \geq a. \end{cases}$



(piecewise continuous)

Now that we have some tools under our belt, here are some of the commonly used transforms.

Commonly Used Transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
$1 = t^0$	$\frac{1}{s} = \frac{0!}{s^{0+1}}$	$s > 0$
t	$\frac{1}{s^2} = \frac{1!}{s^{1+1}}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^a \ (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$u_a(t) = u_0(t-a)$	$\frac{e^{-as}}{s}$	$s > 0$
$\cos kt$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$s > 0$
$\cosh kt$	$\frac{s}{s^2-k^2}$	$s > \|k\|$
$\sinh kt$	$\frac{k}{s^2-k^2}$	$s > \|k\|$

where $n \in \{0,1,2,\ldots\}$, and $a,s,t,k \in \mathbb{R}$.

I won't show you the proof of the above transforms, but you can check the book, or attempt some on your own to verify these.

For a random $f(t)$, does $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$ exist?

Recall (from calculus) that the proper integral $\int_a^b g(t)dt$ exists if g is piecewise continuous on $[a,b]$.

And since e^{-st} is continuous, then if $f(t)$ is piecewise continuous for $t \geq 0$, we have $\int_0^b e^{-st}f(t)dt$ exists for all $b < \infty$.

Here's the hard part: does $\int_0^b e^{-st}f(t)dt$ exist when $b \to \infty$?

In calculus class, you would have just attempted to take the limit of the antiderivative, and determined it that way. But it's not always possible to find an antiderivative. So, let's put a constraint on f to ensure convergence.

A function $f(t)$ is of *exponential order* when there exists:

- nonnegative constants M, c, and T such that $|f(t)| \leq Me^{ct}$ for $T \leq t$.
- In other words, $f(t)$ is of *exponential order* if f is *eventually* smaller than some exponential function.

Existence Criteria Thm: If f is piecewise continuous for $t \geq 0$ and is of exponential order,
then $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ exists for all $s > c$.

Examples of functions that are of exponential order: all **bounded functions** (e.g., trig) & **polynomials**.

Proof: We need to show that $\int_0^\infty e^{-st} f(t)dt$ is a finite number for $s > c$.

But by a standard thm on convergence of improper integrals (absolute convergence implies convergence),
it suffices for us to prove that $\int_0^\infty |e^{-st} f(t)|dt < \infty$ for $s > c$.

Note, $\int_0^\infty |e^{-st} f(t)|dt \leq \int_0^T |e^{-st} M e^{ct}|dt + \int_T^\infty |e^{-st} M e^{ct}|dt$. (replaced $|f(t)|$ and broke up the domain)
But "exponential order" only gives us information when $t \geq T$.

So, before we tackle this, let's find a way of justifying letting $T = 0$. This will make our job easier.

Note that $|f(t)|$ is bounded on $[0, T]$ (due to piecewise continuity).

Also, note that whatever the value of M is, increasing should only make our job harder.
So if we can still show that our integral is bounded after increasing M,
we will still have our proof for the lower value of M.

Thus, let's increase the value of M from whatever it is so that $|f(t)| \leq M$ if $0 \leq t \leq T$ (allowed since $|f(t)|$ is bounded).

Now we see that the first integral term above is just the integral of two bounded functions multiplied together over a finite interval, and therefore finite. So, nothing on the interval $[0, T]$ contributes to our integral being unbounded, and we can therefore evaluate my integral on $t \geq 0$.

So now we have: $\int_0^\infty |e^{-st} f(t)|dt \leq \int_0^\infty |e^{-st} M e^{ct}|dt$

$= M\int_0^\infty e^{-(s-c)t} dt$

$= M[-\frac{1}{s-c} e^{-(s-c)t}]_0^\infty = \frac{M}{s-c} < \infty$ when $s > c$.

Thus, $\int_0^\infty e^{-st} f(t)dt < \infty$. It therefore exists for $s > c$. ■

If we're to use the Laplace transform to solve DEQs, It's vital to know that the resulting soln is uniquely determined.
That is, that the function of s we have found has only one inverse Laplace transform that could be the desired soln.

Uniqueness of Inverse Laplace Transforms: Let f, g be piecewise continuous for $t \geq 0$, and of exponential order as $t \to \infty$ (so that their Laplace transforms $F(s)$ and $G(s)$ exist). If $F(s) = G(s)$ for all $s > c$ (for some c), then on $[0, \infty)$, wherever both f and g are continuous, we have $f(t) = g(t)$.

So we see that a Laplace transform $F(s)$ cannot have originated from two distinct functions (except possibly on some discrete pts).

Examples

Example: Apply the definition $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ to directly find the Laplace transform of $f(t) = \sin^2 t$.

Solution: $\mathcal{L}\{\sin^2 t\} = \int_0^\infty e^{-st} \sin^2 t dt =$

$$= \tfrac{1}{2} \int_0^\infty e^{-st}(1 - \cos 2t)dt = \tfrac{1}{2} \int_0^\infty e^{-st}dt - \tfrac{1}{2} \int_0^\infty e^{-st} \cos 2t dt.$$

Let's first take a look at that second integral…

$\int_0^\infty e^{-st} \cos 2t dt = [-\tfrac{1}{s} e^{-st} \cos 2t]_{t=0}^\infty - \tfrac{2}{s} \int_0^\infty e^{-st} \sin 2t dt$ (IBPs!)

$$= [-e^{-st} \tfrac{\cos 2t}{s}]_{t=0}^\infty - \tfrac{2}{s}\left([-e^{-st} \tfrac{\sin 2t}{s}]_{t=0}^\infty + \tfrac{2}{s} \int_0^\infty e^{-st} \cos 2t dt\right)$$

$$= [-e^{-st} \tfrac{\cos 2t}{s}]_{t=0}^\infty + \left[e^{-st} \tfrac{2 \sin 2t}{s^2}\right]_{t=0}^\infty - \tfrac{4}{s^2} \int_0^\infty e^{-st} \cos 2t dt.$$

$$\left(1 + \tfrac{4}{s^2}\right) \int_0^\infty e^{-st} \cos 2t dt = \left[e^{-st} \tfrac{2 \sin 2t}{s^2} - e^{-st} \tfrac{\cos 2t}{s}\right]_{t=0}^\infty = (0 - 0) - \left(\tfrac{0}{s^2} - \tfrac{1}{s}\right) = \tfrac{1}{s}.$$

So, $\int_0^\infty e^{-st} \cos 2t dt = \dfrac{\frac{1}{s}}{\left(1+\frac{4}{s^2}\right)} = \dfrac{s}{s^2+4}$.

Recall that we were trying to solve $\mathcal{L}\{\sin^2 t\} = \tfrac{1}{2} \int_0^\infty e^{-st}dt - \tfrac{1}{2} \int_0^\infty e^{-st} \cos 2t dt$.

Therefore, $\mathcal{L}\{\sin^2 t\} = \tfrac{1}{2}[-\tfrac{1}{s} e^{-st}]_{t=0}^\infty - \tfrac{1}{2}\left(\tfrac{s}{s^2+4}\right) = \tfrac{1}{2}\left[(0 + \tfrac{1}{s}) - \tfrac{s}{s^2+4}\right] = \tfrac{1}{2}\left[\tfrac{1}{s} - \tfrac{s}{s^2+4}\right].$ □

Example: Use the common transforms in the table to find the Laplace transform of $f(t) = t^{\frac{5}{2}} - e^{-6t}$.

$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\left\{t^{\frac{5}{2}} - e^{-6t}\right\} = \mathcal{L}\left\{t^{\frac{5}{2}}\right\} - \mathcal{L}\{e^{-6t}\}$ (linearity)

$= \dfrac{\Gamma\left(\frac{5}{2}+1\right)}{s^{\frac{5}{2}+1}} - \dfrac{1}{s-(-6)}$. Since

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
t^a $(a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$

.

Note that, $\Gamma\left(\tfrac{5}{2} + 1\right) = \tfrac{5}{2}\Gamma\left(\tfrac{5}{2}\right) = \tfrac{5}{2}\tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{15}{8}\sqrt{\pi}$, since "$\Gamma(n+2) = (n+1)\Gamma(n+1) = n(n+1)\Gamma(n)$," and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

So, $F(s) = \frac{15\sqrt{\pi}}{8s^{\frac{7}{2}}} - \frac{1}{s+6}$. $\square$

Example: What is the inverse Laplace transform of the function $\frac{3}{s^2+16} - \frac{s}{s^2+49}$?

Solution: $\mathcal{L}^{-1}\left\{\frac{3}{s^2+16} - \frac{s}{s^2+49}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+49}\right\}$. (linearity of inverse $\mathcal{L}^{-1}$)

From the table:

$\cos kt$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$s > 0$

So, $\mathcal{L}^{-1}\left\{\frac{3}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+49}\right\} = \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+49}\right\} = \frac{3}{4}\sin 4t + \cos 7t$. $\square$

Exercises 10.1

What did we learn?

- Laplace transforms
- Gamma Functions
- Linearity of Transforms
- Inverse Laplace Transform
- Commonly Used Transforms
- Piecewise Continuous Functions
- Existence and Uniqueness of Laplace Transforms