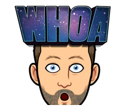
Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Euler Method for Systems

8.2: Non-Homogeneous Linear DEQs



We solved non-homogeneous linear DEQs in §5.5 w/"undetermined coefficients" and "variation of parameters."
Today, we learn to solve *systems* of non-homogeneous DEQs using the same techniques!

Consider a non-homogeneous 1st-order linear system: $\vec{x}' = \mathbf{A}\vec{x} + \vec{f}(t)$, where $\mathbf{A}^{n\times n}$ is constant, and $\vec{f}(t)$ is continuous.

From §7.2, we learned there is a general soln: $\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$, where

- $\vec{x}_c(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t)$ is a general soln of the *associated* homogeneous system: $\vec{x}' = \mathbf{A}\vec{x}$, and
- $\vec{x}_p(t)$ is a single particular soln of the *original* non-homogeneous system.

We have already seen how to calculate $\vec{x}_c(t)$. So this section will help us locate $\vec{x}_p(t)$.

Undetermined Coefficients of Systems

Similar to chapter 5, we make the assumption that $\vec{f}(t)$ is actually a linear combination of polynomials, exponential functions, and sin/cos.

Then, we make an intelligent guess as to the general form of a particular soln, then try to determine the undetermined coefficients in $\vec{x}_p$, by substituting that guess into our DEQ.

Example (polynomial): Find a particular soln of $\vec{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\vec{x} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$.

(assume the pre-trial you find is linearly independent from the homogenous sols)

Solution: Observe that $\vec{f}(t) = \begin{bmatrix} 3 \\ 2t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}t$ is polynomial, so let's try a polynomial trial:

$$\vec{x}_{trial}(t) = \vec{a}t + \vec{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

(we are assuming linear independence from the homogeneous sols)

Substituting into our DEQ:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix} = \begin{bmatrix} 3(a_1 t + b_1) + 2(a_2 t + b_2) + 3 \\ 7(a_1 t + b_1) + 5(a_2 t + b_2) + 2t \end{bmatrix}$$

$$= \begin{bmatrix} (3a_1 t + 3b_1) + (2a_2 t + 2b_2) + 3 \\ (7a_1 t + 7b_1) + (5a_2 t + 5b_2) + 2t \end{bmatrix} = \begin{bmatrix} (3a_1 + 2a_2)t + (3b_1 + 2b_2) \\ (7a_1 + 5a_2 + 2)t + (7b_1 + 5b_2) \end{bmatrix}.$$

So, $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3a_1 + 2a_2 \\ 7a_1 + 5a_2 + 2 \end{bmatrix} t + \begin{bmatrix} 3b_1 + 2b_2 + 3 \\ 7b_1 + 5b_2 \end{bmatrix}.$

Equating powers of t, we find:

$$3a_1 + 2a_2 = 0,$$
$$7a_1 + 5a_2 + 2 = 0,$$
$$3b_1 + 2b_2 + 3 = a_1,$$
$$7b_1 + 5b_2 = a_2.$$

The first two eqs give: $a_1 = 4,\ a_2 = -6.$

Using these in the next two equations give: $b_1 = 17$, $b_2 = -25$. Substituting these into our trial,

we have: $\vec{x}_p(t) = \begin{bmatrix} 4 \\ -6 \end{bmatrix} t + \begin{bmatrix} 17 \\ -25 \end{bmatrix}.$ □

❗ We can check our work! Does $\vec{x}_p(t)$ satisfy the DEQ: $\vec{x}_p' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \vec{x}_p + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$?

On the RHS: $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \left(\begin{bmatrix} 4 \\ -6 \end{bmatrix} t + \begin{bmatrix} 17 \\ -25 \end{bmatrix} \right) + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$

$$= \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix} t + \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 17 \\ -25 \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2 \end{bmatrix} t + \begin{bmatrix} 1 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}.$$

On the LHS: $\vec{x}_p' = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$, so they match! ✓

Example (sinusoidal): Find a particular soln of $\vec{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos t \\ -\cos t - 2\sin t \end{bmatrix}.$

(assume the pre-trial you find is linearly independent from the homogenous sols)

Solution: We see the non-homogeneous part consists of sins and cosines:

$$\vec{f}(t) = \begin{bmatrix} \cos t \\ -\cos t - 2\sin t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \sin t.$$

(we are assuming linear independence from the homogeneous sols, so...)

Therefore, we formulate the trial: $\vec{x}_{trial} = \vec{c}_1 \cos t + \vec{c}_2 \sin t$, where $\vec{c}_1 = (c_{11}, c_{12})$ and $\vec{c}_2 = (c_{21}, c_{22})$.

Then, $\vec{x}'_{trial} = -\vec{c}_1 \sin t + \vec{c}_2 \cos t$ for the LHS. And for the RHS: $\mathbf{A}\vec{x}_{trial} = \mathbf{A}\vec{c}_1 \cos t + \mathbf{A}\vec{c}_2 \sin t$.

Putting it together: $-\vec{c}_1 \sin t + \vec{c}_2 \cos t = \mathbf{A}\vec{c}_1 \cos \mathbf{t} + \mathbf{A}\vec{c}_2 \sin t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \sin t.$

Matching coefficients of cos: $\begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3c_{11} + 2c_{12} + 1 \\ 7c_{11} + 5c_{12} - 1 \end{bmatrix}.$

Matching coefficients of sin: $\begin{bmatrix} -c_{11} \\ -c_{12} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 3c_{21} + 2c_{22} \\ 7c_{21} + 5c_{22} - 2 \end{bmatrix}.$

Substituting c_{21}/c_{22} from the cos system into the sin system:

$$\begin{bmatrix} -c_{11} \\ -c_{12} \end{bmatrix} = \begin{bmatrix} 3(3c_{11} + 2c_{12} + 1) + 2(7c_{11} + 5c_{12} - 1) \\ 7(3c_{11} + 2c_{12} + 1) + 5(7c_{11} + 5c_{12} - 1) - 2 \end{bmatrix} = \begin{bmatrix} 23c_{11} + 16c_{12} + 1 \\ 56c_{11} + 39c_{12} \end{bmatrix}.$$

So, using the first component: $-c_{11} = 23c_{11} + 16c_{12} + 1 \quad \Rightarrow \quad c_{11} = -\frac{16c_{12}+1}{24}.$

From the second component: $-c_{12} = 56c_{11} + 39c_{12} = 56\left(-\frac{16c_{12}+1}{24}\right) + 39c_{12} = \frac{5}{3}c_{12} - \frac{7}{3} \quad \Rightarrow \quad c_{12} = \frac{7}{8}.$

Substituting back into the first component: $c_{11} = -\frac{16\left(\frac{7}{8}\right)+1}{24} = -\frac{5}{8}.$

Substituting back into the first cos system: $\begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} = \begin{bmatrix} 3\left(-\frac{5}{8}\right) + 2\left(\frac{7}{8}\right) + 1 \\ 7\left(-\frac{5}{8}\right) + 5\left(\frac{7}{8}\right) - 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{8} \\ -1 \end{bmatrix}.$

Substituting these into our trial, we have: $\vec{x}_p(t) = \begin{bmatrix} -\frac{5}{8} \\ \frac{7}{8} \end{bmatrix} \cos t + \begin{bmatrix} \frac{7}{8} \\ -1 \end{bmatrix} \sin t.$ ☐

Variation of Parameters of Systems

Similar to §5.5, we can use variation of parameters for systems of DEQs: $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$.

And similar to the single DEQ version, this can be applied to linear DEQs with variable coefficients $\mathbf{P}(t)$, and the non-homogeneous term $\vec{f}(t)$ isn't restricted to polynomials, exponentials, and sinusoidals.

Justification

Assume we have already found the general soln: $\vec{x}_c(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t)$ of the associated homogeneous: $\vec{x}' = \mathbf{P}(t)\vec{x}$.

Note that we can rewrite the general soln as: $\vec{x}_c(t) = \mathbf{X}(t)\vec{c}$, where $\mathbf{X} := \begin{bmatrix} \vec{x}_1 & \ldots & \vec{x}_n \end{bmatrix}$ and $\vec{c} := \begin{pmatrix} c_1 & \ldots & c_n \end{pmatrix}$.

Similar to the justification given in §5.5, in order to accommodate $\vec{f}(t)$ in the non-homogeneous system, we replace the constant $\vec{c}$ w/a vector function $\vec{u}(t)$. So, $\vec{x}_p(t) = \mathbf{X}(t)\vec{u}(t)$. $\quad (*)$

So our task becomes solving for $\vec{u}(t)$. And since $\vec{x}_p(t)$ represents a soln, it must satisfy our non-homogeneous system. So, we take a derivative, and substitute it into: $\vec{x}\,' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$.

$$\vec{x}_p'(t) = \mathbf{X}'(t)\vec{u}(t) + \mathbf{X}(t)\vec{u}\,'(t).$$

Substituting: $\mathbf{X}'(t)\vec{u}(t) + \mathbf{X}(t)\vec{u}\,'(t) = \mathbf{P}(t)\mathbf{X}(t)\vec{u}(t) + \vec{f}(t)$. $\quad (**)$

This looks like a mess, but recall that every soln $\vec{x}_i(t)$ to the homogeneous DEQ must satisfy: $\vec{x}_i' = \mathbf{P}(t)\vec{x}_i$.

Therefore, $\mathbf{X}(t)$, whose column vectors are exactly these type of sols satisfies:
$[\vec{x}_1' \; \ldots \; \vec{x}_n'] = [\mathbf{P}(t)\vec{x}_1 \; \ldots \; \mathbf{P}(t)\vec{x}_n]$ or $\mathbf{X}'(t) = \mathbf{P}(t)\mathbf{X}(t)$.

Substituting this into the LHS of $(**)$, we have: $\mathbf{P}(t)\mathbf{X}(t)\vec{u}(t) + \mathbf{X}(t)\vec{u}\,'(t) = \mathbf{P}(t)\mathbf{X}(t)\vec{u}(t) + \vec{f}(t)$

and upon canceling the common term from both sides we have: $\mathbf{X}(t)\vec{u}\,'(t) = \vec{f}(t)$.

Recall our goal was to solve for $\vec{u}(t)$. So, isolating $\vec{u}\,'$, and integrating gives:
$\vec{u}(t) = \int_0^t \mathbf{X}(t)^{-1}\vec{f}(t)dt$.

Substituting this into $(*)$, gives us our soln, and a new theorem.

Variation of Parameters for Systems Thm: If $\mathbf{X}(t)$ is a fundamental matrix (its colomn vectors contain a full set of sols) for $\vec{x}\,' = \mathbf{P}(t)\vec{x}$ on some interval where $\mathbf{P}(t)$ and $\vec{f}(t)$ are continuous, then a particular soln of the non-homogeneous $\vec{x}\,' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$ is: $\vec{x}_p(t) = \mathbf{X}(t)\int_0^t \mathbf{X}(t)^{-1}\vec{f}(t)dt$.

Adding this to our complementary soln, we get a general soln: $\vec{x}(t) = \mathbf{X}(t)\vec{c} + \mathbf{X}(t)\int_0^t \mathbf{X}(t)^{-1}\vec{f}(t)dt$.

Example: Solve the IVP: $\vec{x}\,' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\vec{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix}te^{-2t}$, $\vec{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, where the soln of the associated homogeneous system gives the fundamental matrix: $\mathbf{X}(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$.

Solution: Recall from above, our general solution will be: $\vec{x}(t) = \mathbf{X}(t)\vec{c} + \mathbf{X}(t)\int_0^t \mathbf{X}(t)^{-1}\vec{f}(t)dt$

Focusing on the difficult integral, note:

$$\mathbf{X}(t)^{-1} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}^{-1} = \tfrac{1}{7}e^{-3t}\begin{bmatrix} e^{5t} & -2e^{5t} \\ 3e^{-2t} & e^{-2t} \end{bmatrix}.$$

And $\mathbf{X}(t)^{-1}\vec{f}(t) = \tfrac{1}{7}e^{-3t}\begin{bmatrix} e^{5t} & -2e^{5t} \\ 3e^{-2t} & e^{-2t} \end{bmatrix}\begin{bmatrix} -15te^{-2t} \\ -4te^{-2t} \end{bmatrix} = \begin{bmatrix} -t \\ -7te^{-7t} \end{bmatrix}.$

Integrating: $\int_0^t \begin{bmatrix} -t \\ -7te^{-7t} \end{bmatrix} dt = \begin{bmatrix} -\int_0^t tdt \\ -7\int_0^t te^{-7t}dt \end{bmatrix} = \begin{bmatrix} -\left[\frac{1}{2}t^2\right]_0^t \\ -7\left(-\frac{1}{7}te^{-7t}|_0^t - \int_0^t(-\frac{1}{7}e^{-7t})dt\right) \end{bmatrix}$ (integration by parts!)

$$= \begin{bmatrix} -\frac{t^2}{2} \\ te^{-7t} - \left[-\frac{1}{7}e^{-7t}\right]_0^t \end{bmatrix} = \begin{bmatrix} -\frac{t^2}{2} \\ te^{-7t} + \frac{1}{7}e^{-7t} - \frac{1}{7} \end{bmatrix}.$$

So: $\vec{x}_p(t) = \mathbf{X}(t)\int_0^t \mathbf{X}(t)^{-1}\vec{f}(t)dt = \mathbf{X}(t)\begin{bmatrix} -\frac{t^2}{2} \\ te^{-7t} + \frac{1}{7}e^{-7t} - \frac{1}{7} \end{bmatrix}.$

Observe for the init-conds, the integral when $t = 0$ would give us zero, so we have: $\vec{x}(0) = \mathbf{X}(0)\vec{c} + \vec{0}$.

Note that $\mathbf{X}(0) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$, so $\begin{bmatrix} 7 \\ 3 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and $c_1 + 2c_2 = 7$ and $-3c_1 + c_2 = 3$.

Solving the first eq: $c_1 = 7 - 2c_2$, and substituting into the 2nd: $-3(7 - 2c_2) + c_2 = 3$.

Solving this for c_2: $6c_2 + c_2 = 3 + 21 \quad \Rightarrow \quad c_2 = \frac{24}{7}$ and $c_1 = 7 - 2(\frac{24}{7}) = \frac{1}{7}$.

So our final answer is: $\vec{x}(t) = \mathbf{X}(t)\vec{c} + \mathbf{X}(t)\int_0^t \mathbf{X}(t)^{-1}\vec{f}(t)dt$

$$= \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}\begin{bmatrix} \frac{1}{7} \\ \frac{24}{7} \end{bmatrix} + \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}\begin{bmatrix} -\frac{t^2}{2} \\ te^{-7t} + \frac{1}{7}e^{-7t} - \frac{1}{7} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7}e^{-2t} + \frac{48}{7}e^{5t} \\ \frac{24}{7}e^{5t} - \frac{3}{7}e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{5t}(\frac{2}{7}e^{-7t} + 2te^{-7t} - \frac{2}{7}) - \frac{1}{2}t^2e^{-2t} \\ \frac{3}{2}t^2e^{-2t} - e^{5t}(\frac{1}{7}e^{-7t} + te^{-7t} - \frac{1}{7}) \end{bmatrix}$$

Thus, $\vec{x}(t) = \frac{1}{14}\begin{bmatrix} (6 - 7t^2 + 28t)e^{-2t} + 92e^{5t} \\ (-4 + 14t + 21t^2)e^{-2t} + 46e^{5t} \end{bmatrix}.$ □

Exercises 8.2 

What did we learn?

- Undetermined Coefficients of Systems
- Variation of Parameters of Systems