

7.6: Multiple Eigenvalue Solutions: Review

To find the complete general solution to $\vec{x}' = \mathbf{A}\vec{x}$, you need n linearly independent vectors from \mathbf{A} 's eigenvalues. But what if one of your eigenvalues λ , has multiplicity k , but only has $k - 1$ eigenvectors (so defect $d = 1$) $\{\vec{u}_1, \dots, \vec{u}_{k-1}\}$. What to do?!?

Algorithm for Eigenvalue of Defect $d = 1$, Multiplicity k

(for example, $\mathbf{A}^{2 \times 2}$, with $\lambda = 5$, multiplicity 2, but only one eigenvector)

◆ Compute: $(\mathbf{A} - \lambda\mathbf{I})^2$.

◆ Choose a nonzero \vec{v}_2 so that: $(\mathbf{A} - \lambda\mathbf{I})^2\vec{v}_2 = \vec{0}$.

often in the book $(\mathbf{A} - \lambda\mathbf{I})^2 = \mathbf{0}$, is the zero matrix. Thus, any nonzero vector \vec{v}_2 will work, so it's traditional to choose a standard vector $\vec{v}_2 = [1 \ 0]^T$ or $\vec{v}_2 = [0 \ 1]^T$.)

◆ Next, calculate $(\mathbf{A} - \lambda\mathbf{I})\vec{v}_2$ and label this as \vec{v}_1 .

◆ If $\vec{v}_1 = \vec{0}$, choose a different nonzero \vec{v}_2 above, and recalculate until your \vec{v}_1 is nonzero.

◆ Then, form the k independent solutions:

$\vec{u}_1 e^{\lambda t}, \dots, \vec{u}_{k-1} e^{\lambda t}$ and $(\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$. (the last one is the "generalized" eigenvector)

Including the rest of the general solution (for the other eigenvalues): $c_1 \vec{w}_1 e^{\lambda_1 t} + \dots + c_{n-k} \vec{w}_{n-k} e^{\lambda_{n-k} t}$, we end up with:

$$x(t) = c_1 \vec{w}_1 e^{\lambda_1 t} + \dots + c_{n-k} \vec{w}_{n-k} e^{\lambda_{n-k} t} + [c_{n-k+1} \vec{u}_1 + \dots + c_{n-1} \vec{u}_{k-1} + c_n (\vec{v}_1 t + \vec{v}_2)] e^{\lambda t}.$$

Most common case: λ with $k = 2$

◆ Form the two independent solutions:

$$\vec{u}_1 e^{\lambda t} \text{ and } (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}.$$

So, from the eigenvalue λ , the contribution to the solution is: $c_1 \vec{u}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$.

General Algorithm for Eigenvalue of Defect d , Multiplicity k

(for example, $\mathbf{A}^{3 \times 3}$, with $\lambda = 5$, multiplicity 3, but only one eigenvector, so $d = 2$)

◆ Compute: $(\mathbf{A} - \lambda\mathbf{I})^{d+1}$.

◆ Choose a nonzero \vec{u}_1 so that: $(\mathbf{A} - \lambda\mathbf{I})^{d+1} \vec{u}_1 = \vec{0}$.

◆ Successively multiply \vec{u}_1 by powers of $(\mathbf{A} - \lambda\mathbf{I})$ until the zero vector is obtained.

◆ When we first form $(\mathbf{A} - \lambda\mathbf{I})^p \vec{u}_1 = \vec{0}$, if the power p is equal to $d + 1$, then you now have vectors $\vec{u}_1, \dots, \vec{u}_{d+1}$ such that:

$$(\mathbf{A} - \lambda\mathbf{I})\vec{u}_1 = \vec{u}_2 \neq \vec{0},$$

⋮

$$(\mathbf{A} - \lambda\mathbf{I})\vec{u}_d = \vec{u}_{d+1} \neq \vec{0}, \text{ and } (\mathbf{A} - \lambda\mathbf{I})\vec{u}_{d+1} = \vec{0}.$$

These k vectors are called your generalized eigenvectors.

◆ If $p < d + 1$, then choose a different nonzero \vec{u}_1 above until you find $d + 1$ nonzero vectors.

◆ Next, form the $d + 1$ independent solutions from:

$$x_1(t) = \vec{u}_{d+1} e^{\lambda t},$$

$$x_2(t) = (\vec{u}_{d+1} t + \vec{u}_d) e^{\lambda t}.$$

⋮

$$x_{d+1}(t) = \left(\vec{u}_{d+1} \frac{t^d}{d!} + \dots + \vec{u}_3 \frac{t^2}{2!} + \vec{u}_2 t + \vec{u}_1 \right) e^{\lambda t}.$$

Observe that if our defect has $d + 1 < k$, the above algorithm does not produce k linearly independent eigenvectors/solutions. However, if $d + 1 < k$, this means that we have additional independent vectors found through the normal process. For example, if we are working with a 4×4 matrix with one eigenvalue λ , and find 2 ordinary vectors \vec{v}_1, \vec{v}_2 , we have a defect of 2, so $d + 1 = 3 < k = 4$. However, we have the 2 ordinary vectors which are linearly independent, in addition to the 3 vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ we find above using our new process. So obviously all 5 of these cannot be linearly independent. In fact, we will also find that \vec{u}_1 is a linear combination of \vec{v}_1, \vec{v}_2 . So, the way we find 4 **linearly independent** vectors here is to find a linear combination of \vec{v}_1, \vec{v}_2 (which we will call \vec{u}_4) that is linearly independent from \vec{u}_1 . Then, our basis becomes $\{\vec{u}_1 e^{\lambda t}, (\vec{u}_2 t + \vec{u}_1) e^{\lambda t}, (\vec{u}_3 t^2 + \vec{u}_2 t + \vec{u}_1) e^{\lambda t}, \vec{u}_4 e^{\lambda t}\}$.

Including the rest of the general solution (for the other eigenvalues): $c_1 \vec{w}_1 e^{\lambda_1 t} + \dots + c_{n-k} \vec{w}_{n-k} e^{\lambda_{n-k} t}$, we end up

with: $x(t) = c_1 \vec{w}_1 e^{\lambda_1 t} + \dots + c_{n-k} \vec{w}_{n-k} e^{\lambda_{n-k} t}$
 $+ [c_{n-k+1} \vec{u}_{d+1} + c_{n-k+2} (\vec{u}_{d+1} t + \vec{u}_d) + \dots + c_{n-k+d-2} (\vec{u}_{d+1} \frac{t^d}{d!} + \dots + \vec{u}_3 \frac{t^2}{2!} + \vec{u}_2 t + \vec{u}_1)] e^{\lambda t}$
 $+ [c_{n-k+d-1} \vec{u}_{d+2} + \dots + c_n \vec{u}_k] e^{\lambda t}.$

Problem: #6 Find a general solution to the system: $\vec{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \vec{x}.$

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = (1-\lambda)(9-\lambda) + 16 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

$$\lambda_1 = \lambda_2 = 5$$

$$\begin{bmatrix} 1-5 & -4 \\ 4 & 9-5 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = b \text{ and } x = -b.$$

$$\vec{u}_1 = [-1 \ 1]^T$$

Only one eigenvector? Defective! $d = 1.$

"Compute: $(\mathbf{A} - \lambda \mathbf{I})^2.$ "

$$(\mathbf{A} - \lambda \mathbf{I})^2 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

"Choose a nonzero vector \vec{v}_2 so that: $(\mathbf{A} - \lambda \mathbf{I})^2 \vec{v}_2 = \vec{0}.$ "

$(\mathbf{A} - \lambda \mathbf{I})^2 \vec{v}_2 = \vec{0}$ is therefore satisfied by *any* choice of $\vec{v}_2.$

Generic nonzero choice in these situations: $\vec{v}_2 = [1 \ 0]^T.$

"Calculate $(\mathbf{A} - \lambda \mathbf{I}) \vec{v}_2$ and label this $\vec{v}_1.$ "

$$(\mathbf{A} - \lambda \mathbf{I}) \vec{v}_2 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} =: \vec{v}_1 \quad (\text{yay, nonzero!})$$

"Form the two independent solutions: $\vec{x}_1(t) = \vec{u}_1 e^{\lambda t}$ and $\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}.$ "

$$\text{So, } \vec{x}_1(t) = \vec{u}_1 e^{5t} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}.$$

$$\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{5t} = \left(\begin{bmatrix} -4 \\ 4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{5t} = \begin{bmatrix} -4t + 1 \\ 4t \end{bmatrix} e^{5t}.$$

Gen. solution: $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -4t + 1 \\ 4t \end{bmatrix} e^{5t}$

$$= \begin{bmatrix} -4c_1 + (-4t + 1)c_2 \\ 4c_1 + 4tc_2 \end{bmatrix} e^{5t}.$$

Problem: #16 Find a general solution to the system: $\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix} \vec{x}$.

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\begin{vmatrix} 1 - \lambda & 0 & 0 \\ -2 & -2 - \lambda & -3 \\ 2 & 3 & 4 - \lambda \end{vmatrix} = (1 - \lambda)((-2 - \lambda)(4 - \lambda) + 9) = (1 - \lambda)(\lambda^2 - 2\lambda + 1)$$

$$= -(\lambda - 1)^3 = 0. \quad \text{OR}$$

$$\begin{vmatrix} 1 - \lambda & 0 & 0 \\ -2 & -2 - \lambda & -3 \\ 2 & 3 & 4 - \lambda \end{vmatrix} = \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 - \lambda \\ 2 & 3 & 4 - \lambda \end{vmatrix} = \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ 2 & 3 & 1 - \lambda \end{vmatrix}$$

$$= (\lambda - 1)^3 = 0.$$

So: $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

Now: $(\mathbf{A} - \lambda \mathbf{I})\vec{v} = \vec{0}$.

Reductions: $\begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{3}{2} & \frac{3}{2} \end{bmatrix}, \quad z = c, y = b, x = -\frac{3}{2}b - \frac{3}{2}c$

So, $\langle -\frac{3}{2}b - \frac{3}{2}c, b, c \rangle = \langle 3, -2, 0 \rangle + \langle 3, 0, -2 \rangle$, where $b = c = -2$.

$\vec{u}_1 = \langle 3, -2, 0 \rangle$, and $\vec{u}_2 = \langle 3, 0, -2 \rangle$.

Only two eigenvectors? Defective! $d = 1$.

$$(\mathbf{A} - \lambda \mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

"Choose a nonzero vector \vec{v}_2 so that: $(\mathbf{A} - \lambda \mathbf{I})^2 \vec{v}_2 = \vec{0}$."

Start with generic: $\vec{v}_2 = [1 \ 0 \ 0]^T$.

Calculate $\vec{v}_1 := (\mathbf{A} - \lambda \mathbf{I})\vec{v}_2$.

$$\vec{v}_1 := (\mathbf{A} - \lambda\mathbf{I})\vec{v}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}. \quad (\text{yay, nonzero!})$$

Notice that: $(\mathbf{A} - \lambda\mathbf{I})\vec{v}_1 = (\mathbf{A} - \lambda\mathbf{I})((\mathbf{A} - \lambda\mathbf{I})\vec{v}_2) = (\mathbf{A} - \lambda\mathbf{I})^2\vec{v}_2 = \vec{0}$.

So, $\vec{v}_1 = \langle 0, -2, 2 \rangle$ is an ordinary eigenvector associated with λ .

Also, recall: $\vec{u}_1 = \langle 3, -2, 0 \rangle$ and $\vec{u}_2 = \langle 3, 0, -2 \rangle$ are also eigenvectors associated with λ .

So, we might mistakenly think that $\vec{x}(t) = e^t[c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{v}_1]$ is our general solution.

However, \vec{v}_1 is a linear combination of $\vec{u}_1 = \langle 3, -2, 0 \rangle$ and $\vec{u}_2 = \langle 3, 0, -2 \rangle$, namely: $\vec{u}_1 - \vec{u}_2 = \vec{v}_1$.

So, $\vec{v}_1 e^t$ is a linear combination of the independent solutions $\vec{u}_1 e^t$ and $\vec{u}_2 e^t$ (and therefore dependent). So, we must instead use the prescribed $(\vec{v}_1 t + \vec{v}_2) e^t$ as the desired third **independent** solution.

The corresponding general solution is described by

$$\vec{x}(t) = e^t[c_1\vec{u}_1 + c_2\vec{u}_2 + c_3(\vec{v}_1 t + \vec{v}_2)]$$

OR

$$\vec{x}_1(t) = e^t(3c_1 + 3c_2 + c_3)$$

$$\vec{x}_2(t) = e^t(-2c_1 - 2c_3 t)$$

$$\vec{x}_3(t) = e^t(-2c_2 + 2c_3 t).$$

Problem - Find a general solution to the system: $\vec{x}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} \vec{x}$, where $\lambda \in \{0, -2, -2, -2\}$, and the

eigenvector associated with $\lambda_1 = 0$ is $\vec{u}_1 = [1 \ 1 \ 0 \ 0]^T$.

$$\mathbf{A} - \lambda_2\mathbf{I} = \mathbf{A} + 2\mathbf{I} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \xrightarrow{R_4+R_3 \text{ and } R_3+R_1} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix},$$

$$x_4 = s, \quad x_3 = t, \quad 2x_2 = -s, \quad 2x_1 = -t.$$

$$\Rightarrow \vec{x} = \langle -\frac{1}{2}t, -\frac{1}{2}s, t, s \rangle = \langle 1, 0, -2, 0 \rangle + \langle 0, 1, 0, -2 \rangle, \text{ when } t, s = -2.$$

So: $\vec{u}_2 = \langle 1, 0, -2, 0 \rangle$ and $\vec{u}_3 = \langle 0, 1, 0, -2 \rangle$. Defect $d = 1$.

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

"Choose a nonzero vector \vec{v}_2 so that: $(\mathbf{A} - \lambda\mathbf{I})^2\vec{v}_2 = \vec{0}$."

Start with generic: $[1\ 0\ 0\ 0]^T$.

$$(\mathbf{A} - \lambda\mathbf{I})^2[1\ 0\ 0\ 0]^T = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \neq 0, \text{ so we see this doesn't work.}$$

But looking at $(\mathbf{A} - \lambda\mathbf{I})^2$, and making a more informed choice, we choose $[1\ 0\ -2\ 0]^T$. And indeed:

$$(\mathbf{A} - \lambda\mathbf{I})^2[1\ 0\ -2\ 0]^T = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, our next task is to determine nonzero $\vec{v}_1 := (\mathbf{A} - \lambda\mathbf{I})\vec{v}_2$, but observe that

$$(\mathbf{A} - \lambda\mathbf{I})[1\ 0\ -2\ 0]^T = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ But we need this to be nonzero, so apparently we must}$$

choose a different \vec{v}_2 vector above. By looking at $(\mathbf{A} - \lambda\mathbf{I})^2$ again, and making another informed choice, we choose $\vec{v}_2 = [0\ 0\ 1\ -1]^T$. And notice that

$$(\mathbf{A} - \lambda\mathbf{I})^2[0\ 0\ 1\ -1]^T = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ So that is good. But what about } \vec{v}_1 := (\mathbf{A} - \lambda\mathbf{I})\vec{v}_2 \text{ ?}$$

$$\vec{v}_1 := (\mathbf{A} - \lambda\mathbf{I})\vec{v}_2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}. \quad (\text{yay, nonzero!})$$

The corresponding general solution is described by

$$\vec{x}(t) = c_1\vec{u}_1e^0 + c_2\vec{u}_2e^{-2t} + c_3\vec{u}_3e^{-2t} + c_4(\vec{v}_1t + \vec{v}_2)e^{-2t}$$

OR

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} e^{-2t} + c_4 \left(\begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right) e^{-2t}$$

Observe that $\vec{v}_1 = \vec{u}_2 - \vec{u}_3$.