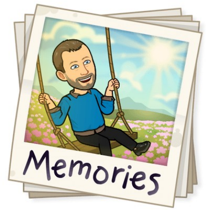
Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Transforming Higher-Order DEQs into a System of 1st-Order DEQs
- Transforming a System of 1st-Order DEQs into a 2nd-Order DEQ
- Existence and Uniqueness of Sols for Linear Systems
- Examples of 2D Systems



7.2: Matrices and Linear Systems

Let's simplify a *system* of DEQs

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + f_1(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + f_n(t) \end{aligned}$$

by expressing it as a *single* matrix DEQ: $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$

Matrices can be Functions! $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ or $\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}$.

For example: $\mathbf{A}(t) = \begin{bmatrix} 0 & -e^{2t} \\ 4e^{3t} & 12t \end{bmatrix}$.

A matrix function is **continuous** or **differentiable** at a pt t (or on an interval $a \le t \le b$) if each of its components $a_{mn}(t)$ are continuous or differentiable there.

And we take the derivative component-wise: $\mathbf{A}'(t) = \frac{d\mathbf{A}}{dt} = \begin{bmatrix} a_{11}'(t) & a_{12}'(t) & a_{13}'(t) \\ a_{21}'(t) & a_{22}'(t) & a_{23}'(t) \\ a_{31}'(t) & a_{32}'(t) & a_{33}'(t) \end{bmatrix}$.

Also, similar to calculus we have:
$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}'\mathbf{B} + \mathbf{AB}'$ (product rule), and $\frac{d}{dt}(\mathbf{CA}) = \mathbf{CA}'$, where $\mathbf{C}$ is a (normal) constant **matrix**.

Notationally Transforming a System

Given: $$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + f_1(t), \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + f_2(t). \end{aligned}$$

Notate: $\vec{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{P}(t) := \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$, $\vec{f}(t) := \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$.

Thus: $\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$.

And the system becomes: $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$.

A **soln to this DEQ** on the open interval I consists of a *column vector function* $\vec{x}(t) = (x_1(t), \ldots, x_n(t))$, such that the component functions of $\vec{x}$ satisfy the system identically on I.

If the p_{ij} and f_i are continuous on I, then the existence and uniqueness thm of the previous section applies for init-conds in I.

Similarities between Sols to Systems and Individual DEQs

As w/our previous *individual* DEQs ($x^{(n)} = f(x^{(n-1)}, \ldots, x, t)$), for a *system* ($\vec{x}' = \mathbf{P}(t)\vec{x}$) we have the following.

Superposition Principal/Gen. Sols Thm:

If $\vec{x}_1(t),\ \vec{x}_2(t),\ \ldots,\ \vec{x}_n(t)$ are (vector) sols of the system $\vec{x}'(t) = \mathbf{P}(t)\vec{x}$, then so is the linear combination: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \ldots + c_n\vec{x}_n(t)$, for *any* choice of c_i.

Proof: We need $\vec{x}' = \mathbf{P}(t)\vec{x}$, where $\vec{x}$ is the linear combonation.

Taking the derivative of $\vec{x}$, we have: $\vec{x}' = c_1\vec{x}_1' + c_2\vec{x}_2' + \ldots + c_n\vec{x}_n'$.

And since we are given that $\vec{x}_i$ are sols, we have $\vec{x}_i' = \mathbf{P}(t)\vec{x}_i$ for each i $(1 \le i \le n)$.

$$\text{So: } \vec{x}' = c_1\mathbf{P}(t)\vec{x}_1 + c_2\mathbf{P}(t)\vec{x}_2 + \ldots + c_n\mathbf{P}(t)\vec{x}_n$$

$$= \mathbf{P}(t)(c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_n\vec{x}_n).$$

That is, $\vec{x}' = \mathbf{P}(t)\vec{x}$, as desired. ■

Independence of Vector Valued Functions: Vector functions $\vec{x}_1, \ldots, \vec{x}_n$ are linearly **dependent** on the interval I provided that there exist constants $c_1, \ldots, c_n$, *not all zero* such that $c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \ldots + c_n\vec{x}_n(t) = \vec{0}$.
As before, they're linearly independent provided none of them is a linear combination of the others (e.g., $\vec{x}_1(t) \neq k\vec{x}_2(t)$).

Wronskian: If $\vec{x}_1(t),\ \vec{x}_2(t),\ \vec{x}_3(t)$ are sols of $\vec{x}' = \mathbf{P}(t)\vec{x}$ on open interval I, and $\mathbf{P}(t)$ is continuous on I, then let $W(\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)) := \det\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix}$ (note that this Wronskian doesn't involve derivatives). Then we have:

- ♦ $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are linearly dependent on I *iff* $W = 0$ at every pt of I.
- ♦ $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are linearly independent on I *iff* $W \neq 0$ at every pt of I.

General Sols of Homogeneous Systems Thm in $\mathbb{R}^n$: Let $\vec{x}_1, \ldots, \vec{x}_n$ be n linearly independent sols of $\vec{x}' = \mathbf{P}(t)\vec{x}$ on an open interval I where $\mathbf{P}(t)$ is continuous. If $\vec{x}(t)$ is any soln whatsoever of the DEQ $\vec{x}' = \mathbf{P}(t)\vec{x}$ on I, then there exist numbers $c_1, \ldots, c_n$ such that: $\vec{x}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t)$, for all t in I.

As a result, general sols can be written as $\vec{x}(t) = \mathbf{X}(t)\vec{c}$, where $\mathbf{X} := \begin{bmatrix}\vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n\end{bmatrix}$, and $\vec{c} := \begin{pmatrix}c_1 & \ldots c_n\end{pmatrix}$.

And solving for init-conds $\vec{x}(a) = \vec{b}$ with $\vec{b} := \begin{pmatrix}b_1 & b_2 & \ldots & b_n\end{pmatrix}$ can be accomplished by substituting

it into the above soln as $\vec{b} = \mathbf{X}(a)\vec{c}$. In order to solve for $c_1, \ldots c_n$.

General Sols of Non-Homogeneous Systems in $\mathbb{R}^n$: For $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$, let $\vec{x}_p$ be a particular soln on open interval I, where $\mathbf{P}(t)$, $\vec{f}(t)$ are continuous. Let $\vec{x}_1, \ldots, \vec{x}_n$ be linearly independent sols of the associated homogeneous DEQ $\vec{x}' = \mathbf{P}(t)\vec{x}$ on I. If $\vec{x}(t)$ is any soln whatsoever of the non-homogeneous DEQ on I, then there exist numbers $c_1, \ldots, c_n$ such that: $\vec{x}(t) = \vec{x}_p(t) + \vec{x}_c(t) = \vec{x}_p + (c_1\vec{x}_1 + \ldots + c_n\vec{x}_n)$.

Example: Given the system of DEQs below, given in matrix form: $x' = \mathbf{A}\vec{x}$. Verify that the given vector functions are sols to that system. Then, use the Wronskian to show that the sols are linearly independent.

Finally, write the general soln to the system: $\vec{x}' = \begin{bmatrix} 4 & -2 & 2 \\ 2 & 0 & 2 \\ -2 & 2 & 0 \end{bmatrix}\vec{x}$.

Possible Sols: $\vec{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{y}_2 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ 0 \end{bmatrix}$, $\vec{y}_3 = \begin{bmatrix} -e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$.

Soln: LHS, $\vec{y}_1' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{y}_2' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \\ 0 \end{bmatrix}$, $\vec{y}_3' = \begin{bmatrix} -2e^{2t} \\ 0 \\ 2e^{2t} \end{bmatrix}$.

RHS: $\begin{bmatrix} 4 & -2 & 2 \\ 2 & 0 & 2 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. $\begin{bmatrix} 4 & -2 & 2 \\ 2 & 0 & 2 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \\ 0 \end{bmatrix}$.

$\begin{bmatrix} 4 & -2 & 2 \\ 2 & 0 & 2 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} -e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} = \begin{bmatrix} -2e^{2t} \\ 0 \\ 2e^{2t} \end{bmatrix}$. ✓ They all match!

Finally, the wronskian is: $W = |\vec{y}_1\, \vec{y}_2\, \vec{y}_3| =$

$$\begin{vmatrix} -1 & e^{2t} & -e^{2t} \\ -1 & e^{2t} & 0 \\ 1 & 0 & e^{2t} \end{vmatrix}$$

$$= e^{2t}e^{2t}\begin{vmatrix} -1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = e^{4t}\begin{vmatrix} -1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$= e^{4t}(-1(0-1)) \ = e^{4t} \neq 0$ for any t. So, $\vec{y}_1$, $\vec{y}_2$, and $\vec{y}_3$ are linearly independent.

“**Finally**, **write the general solution to the system**."

Since we have **three** linearly independent solutions for a system of **three** first order DEQs, we can write the general solution:

$$\vec{x}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 = c_1\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} e^{2t} \\ e^{2t} \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}.$$

OR with different notation: $$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -c_1 + (c_2 - c_3)e^{2t} \\ -c_1 + c_2 e^{2t} \\ c_1 + c_3 e^{2t} \end{bmatrix}.$$

Exercises 7.2 

What did we learn?

- Matrix Functions
- Notationally Transforming a System
- Similarities between Solutions to Systems of DEQs and Individual DEQs