Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- **Transition Matrices**
- **Predator-Prey Models**
- **Cayley-Hamilton Theorem**

7.1: 1st-Order Systems and Applications

We've discussed ways of solving a *single* DEQ involving *one dependent variable* (and it's derivatives):

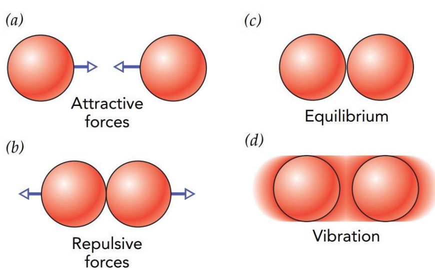
$y' = f(y,t)$, or $x'' = f(x', x, t)$, etc.

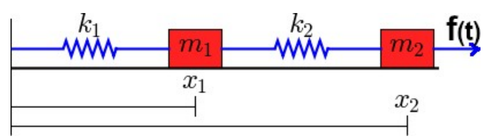
In this chapter, we'll learn to solve *systems* of DEQs involving *several dependent variables*.

In particular, systems which have as many equations as dependent variables.

For example, this model of two (x_1, x_2) vibrating atoms in a molecule:

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1) + f(t)$$





atoms vibrating model

The matrix manipulation skills we learned will help us to solve these types of problems in coming sections.

To simplify these problems, we introduce a technique to reduce the order of a DEQ.

In particular, we have more tools to solve 1st-order than higher-order DEQs.

And it turns out we can transform higher-order into 1st-order DEQs!

Transforming Higher-Order DEQs into Systems of 1st-Order DEQs

If you're given $x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$, then define some new variables:

$$x_0 := x, \qquad x_1 := x' \ (= x_0'), \qquad x_2 := x'' \ (= x_1').$$

(note that you don't need the highest derivative $x^{(3)}$ to be represented by a new variable)

Using this "dictionary," we can define a 1st-order system to replace our given DEQ.

Particularly, the first two DEQs listed below simply come from the dictionary itself.

The last DEQ is the given DEQ, but with the variable x swapped out for the new ones.

$$x_0' = x_1,$$
$$x_1' = x_2,$$
$$x_2' + 3x_2 + 2x_1 - 5x_0 = \sin 2t.$$

(notice that the old variable x is eliminated completely)

Why would we make this transformation? Because 1st-order DEQs are easier to solve, especially for systems of DEQs. The price we pay is that now we have to solve three of them! Which is part of why we learned about matrices.

Transforming a System of 1st-Order DEQs into a 2nd-Order DEQ

Now let's go the other way around, turning a system of 1st-order DEQs into one of higher-order DEQ.

Why would we make this transformation? Observe that the system below is coupled (e.g., there is a y in the x' DEQ). For simple low dimensional coupled systems like this one, instead of using matrix methods (e-vals/e-vecs), it may be simpler to do a transformation (as we do below) and solve the resulting DEQ using the methods learned previously in the course.

Transforming and solving a 2D System: Example, $x' = -2y, \quad y' = \frac{1}{2}x$.

- ♦ Attempt to alter the DEQs toward the aim of substituting one into the other.

- ♦ The resulting DEQs should be either in x or y, depending on your choice above.

  For the example above, you can differentiate the first DEQ, and then substitute the second DEQ into it:

  $$x'' = -2y' = -2\left(\tfrac{1}{2}x\right) = -x, \qquad \Rightarrow \qquad x'' + x = 0.$$

- ♦ Next, use your standard techniques to solve for $x(t)$.

- ♦ Finally, use substitution again to plug $x(t)$ into $x' = -2y$ and solve for $y(t)$.

Existence and Uniqueness of Sols for Linear Systems

Given: $x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + p_{13}(t)x_3 + f_1(t)$,

$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + p_{23}(t)x_3 + f_2(t)$,

$x_3' = p_{31}(t)x_1 + p_{32}(t)x_2 + p_{33}(t)x_3 + f_3(t)$,

w/init-conds: $x_1(a) = b_1, \quad x_2(a) = b_2, \quad x_3(a) = b_3$.

If the p_{ij} and f_k are continuous on some I containing $t = a$,

then there **exists** a **unique** soln $(x_1(t), x_2(t), x_3(t))$ to the system within the interval I. (and similarly for larger DEQ systems)

Examples of 2D Systems

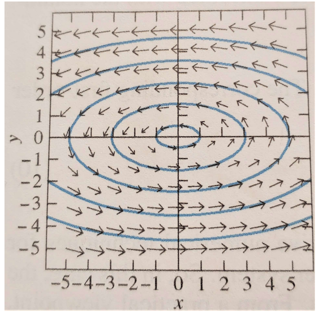
Example: Finishing our example from above: $x' = -2y, \quad y' = \frac{1}{2}x$.

Taking the derivative of the first DEQ: $x'' = -2y' = -2\left(\frac{1}{2}x\right) = -x$.

So: $x'' + x = 0 \quad \Rightarrow \quad r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \quad \Rightarrow \quad e^{it} = \cos t + i \sin t$

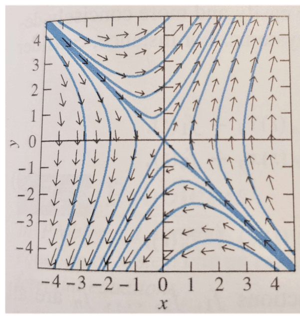
$\Rightarrow \quad x(t) = A\cos t + B\sin t$.

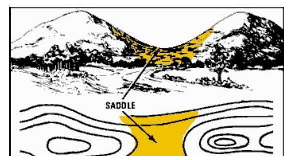
From the first given DEQ, we have: $y = -\frac{1}{2}x' = \frac{1}{2}(A\sin t - B\cos t)$.



planetary orbits, anything cyclic

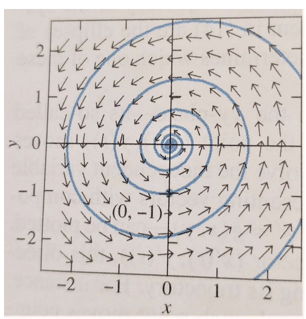
Example: For $x' = y, \quad y' = 2x + y,$ we find soln curves like...

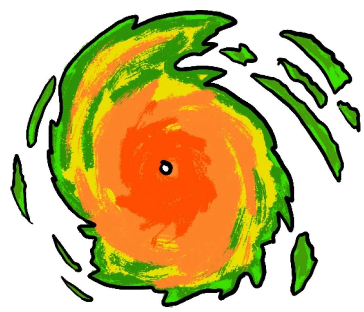




topography

Example: for the following system: $x' = -y, \quad y' = \frac{101}{100}x - \frac{1}{5}y,$ we find soln curves like ...





Hurricanes, fluidic motion, decaying orbit

Why do such similar systems generate such different soln curves?

Stay tuned for an answer in subsequent sections.

Exercises 7.1

What did we learn?

- Transforming Higher-Order DEQs into a System of 1st-Order DEQs
- Transforming a System of 1st-Order DEQs into a 2nd-Order DEQ
- Existence and Uniqueness of Sols for Linear Systems
- Examples of 2D Systems