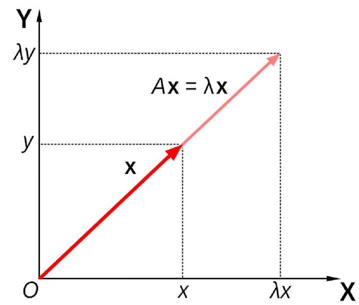
Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Undetermined Coefficients
- Variation of Parameters

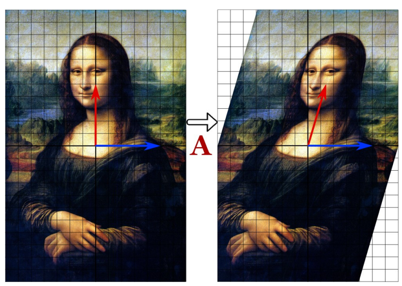
6.1: Eigenvalues



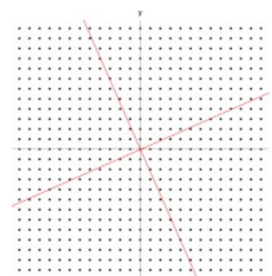
E-vec $\vec{x}$ multiplied by A

Sometimes you're working with a matrix **A**, and you find that, for some vectors $\vec{v}$, that multiplying them by **A** is equivalent to multiplying it by a scalar λ: $\mathbf{A}\vec{v} = \lambda\vec{v}$.

For an $n \times n$ matrix **A**, if we have: $\mathbf{A}\vec{v} = \lambda\vec{v}$, then λ is called an **eigenvalue (e-val)**, and $\vec{v}$ is an **eigenvector (e-vec)**, where $\vec{v}$ is a nonzero vector, and $\lambda \in \mathbb{C}$.
(or equavalently, $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$, where **I** is the identity matrix)



vector (red), eigenvector (blue) under $\mathbf{A}$

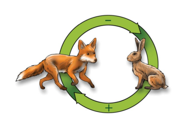


(see animated during class)

Motivation: Recall how pts in the $\mathbb{R}^n$ vector space are characterized as linear combinations of the unit vectors u_x along the axes; $(1,-3,5) = 1(1,0,0) - 3(0,1,0) + 5(0,0,1) = 1u_x - 3u_y + 5u_z$.

Analogously, sols in the soln space to a *system* of DEQs $\vec{x}' = \mathbf{A}\vec{x}$ (we will learn about this in chapter 7) can be characterized as linear combinations of $e^{\lambda t}\vec{v}_\lambda$, where $\lambda, \vec{v}_\lambda$ are the e-vals and e-vecs of **A**.

Applications: Modeling (animal) migration patterns, predator-prey relationships, fluid dynamic, and many more.



Calculation

Given $\mathbf{A}^{n\times n}$, how do we find its e-vals and e-vecs?

We want $\lambda, \vec{v}$ such that: $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$.

Observe that this is a homogeneous system of n equations, where our n unknowns are the components of $\vec{v}$.

Recall that such a system has a nontrivial soln ($\vec{v} \neq 0$) when $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

So, by imposing this restriction, we can identify the various λ.

Characteristic Equation: $|\mathbf{A}-\lambda\mathbf{I}| =$

$$= \left| \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{(n-1)n} \\ a_{n1} & \dots & a_{n(n-1)} & a_{nn}-\lambda \end{vmatrix} = 0.$$

Finding e-vals and e-vecs of **A**:

♦ Solve $|\mathbf{A}-\lambda\mathbf{I}| = c_0\lambda^n + c_1\lambda^{n-1}\dots+c_n = 0$, for all λ_k (should be n e-vals, if you include multiplicity)

♦ Then, for each λ_k, solve $(\mathbf{A}-\lambda_k\mathbf{I})\vec{v} = \vec{0}$ to find the e-vecs $\vec{v}$ for λ_k.

$$(\mathbf{A}-\lambda_k\mathbf{I})\vec{v} = \begin{bmatrix} a_{11}-\lambda_k & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn}-\lambda_k \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Example: Find the (real) e-vals and the associated e-vecs for: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix}$.

$$|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -6 & 8-\lambda & 2 \\ 12 & -15 & -3-\lambda \end{vmatrix} = (1-\lambda)((8-\lambda)(-3-\lambda)-(-30))$$ (pro tip, keep the 1st factor separate; don't turn this into a cubic!)

$$= (1-\lambda)(\lambda^2-5\lambda+6) = (1-\lambda)(\lambda-2)(\lambda-3), \text{ so } \lambda \in \{1,2,3\}.$$

Finding the e-vecs: $(\mathbf{A}-\lambda\mathbf{I})\vec{v} = \vec{0}$.

$$\lambda = 1: \quad \begin{bmatrix} 1-1 & 0 & 0 \\ -6 & 8-1 & 2 \\ 12 & -15 & -3-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -6 & 7 & 2 \\ 12 & -15 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 7 & 2 \\ 12 & -15 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}.$$

Setting $z = s$, we have $y = 0$ and $-6x = -2s$ or $x = \frac{s}{3}$. Thus: $\vec{v}_1 = (x,y,z) = \left(\frac{s}{3},0,s\right)$ or $(1,0,3)$ when $s = 3$.

$$\lambda = 2: \quad \begin{bmatrix} 1-2 & 0 & 0 \\ -6 & 8-2 & 2 \\ 12 & -15 & -3-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -6 & 6 & 2 \\ 12 & -15 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & -15 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $z = s$, we have $3y = -s$ or $y = -\frac{s}{3}$ and $x = 0$. Thus: $\vec{v}_2 = (x,y,z) = \left(0,-\frac{s}{3},s\right)$ or $(0,-1,3)$ when $s = 3$.

$$\lambda = 3: \quad \begin{bmatrix} 1-3 & 0 & 0 \\ -6 & 8-3 & 2 \\ 12 & -15 & -3-3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -6 & 5 & 2 \\ 12 & -15 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & -15 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Setting $z = s$, we have $5y = -2s$ or $y = -\frac{2}{5}s$ and $x = 0$. Thus: $\vec{v}_3 = (x,y,z) = (0,-\frac{2}{5}s,s)$ or $(0,-2,5)$ when $s = 5$.

Complex E-vals: If the components of **A** are real, then any complex e-vals will occur in conjugate pairs (i.e., $\lambda_{\pm} = a \pm bi$).

And so will the e-vecs!! So if the the e-vec of $\lambda_+ = a + bi$ is $\vec{v}_+ = (1+2i,\ -2-3i)$,
then the e-vec of $\lambda_- = a - bi$ is $\vec{v}_- = (1-2i,\ -2+3i)$.

Example: Find the e-vals and the associated e-vecs for: $\mathbf{A} = \begin{bmatrix} 0 & 24 \\ -6 & 0 \end{bmatrix}$.

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 24 \\ -6 & -\lambda \end{vmatrix} = \lambda^2 + 144 = 0 \quad \Rightarrow \quad \lambda = \pm 12i.$$

Finding the e-vecs: $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$.

$$\lambda = 12i: \quad \begin{bmatrix} -12i & 24 \\ -6 & -12i \end{bmatrix} \overset{-\frac{1}{12}R_1 \ \& \ -\frac{1}{6}R_2}{\Rightarrow} \begin{bmatrix} i & -2 \\ 1 & 2i \end{bmatrix} \overset{R_2+iR_1}{\Rightarrow} \begin{bmatrix} i & -2 \\ 0 & 0 \end{bmatrix} \overset{-iR_1}{\Rightarrow} \begin{bmatrix} 1 & 2i \\ 0 & 0 \end{bmatrix}.$$

Setting $y = s$, we have $x = -2is$. Thus: $\vec{v}_1 = (x,y) = (-2is,s)$ or $(-2i,1)$ when $s = 1$.

And since our coefficient matrix **A** had real entries, our e-vals and e-vecs come in conjugate pairs.

And therefore, associated with $\lambda = -12i$, we have: $\vec{v}_2 = (2i,1)$. □

Eigenspaces

Each e-val λ_k associated w/$\mathbf{A}^{n \times n}$ will produce a set of (one or more) linearly independent e-vecs $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_m\}$.

For each λ_k, the associated e-vecs form an basis for a subspace of $\mathbb{R}^n$, $span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_m\} = \mathbb{R}^m \subseteq \mathbb{R}^n$.

This subspace is called an **eigenspace**, and is FULL of e-vecs which are linear combinations of the discovered basis.

The eigenspace of each λ_k serves as the soln space to $(\mathbf{A} - \lambda_k I)\vec{v} = \vec{0}$.

Example: Find the (real) eigenvalues, the associated eigenvectors, and a basis for each eigenspace for $\mathbf{A} = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$$\begin{vmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} \overset{\text{(trust me)}}{=} -(\lambda-1)(\lambda-2)^2 \quad \Rightarrow \quad \lambda \in \{1,2\}, \text{ where } \lambda = 2 \text{ has multiplicity } 2.$$

$$\lambda = 1: \quad \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Setting $y = s$, we have: $z = 0$ and $x = s$. Thus: $\vec{v}_1 = (s, s, 0) = (1, 1, 0)$, where $s = 1$.

And the basis for the $\lambda = 1$ eigenspace is $\{\vec{v}_1\}$.

$$\lambda = 2: \quad \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}. \quad \text{Set } y = s,\ z = t.$$

Thus: $2x = -3s - t$ and $(-3s - t, s, t) = s(-3, 1, 0) + t(-1, 0, 1)$. This gives: $\vec{v}_2 = (-3, 1, 0)$ and $\vec{v}_3 = (-1, 0, 1)$.

And the basis for the $\lambda = 2$ eigenspace is $\{\vec{v}_2, \vec{v}_3\}$.

Video Tutorial (visually rich and intuitive): https://youtu.be/PFDu9oVAE-g

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Exercises 6.1 

What did we learn?

- Eigenvalues, Eigenvectors: $\mathbf{A}\vec{v} = \lambda\vec{v}$
- Characteristic Equation: $|\mathbf{A} - \lambda\mathbf{I}| = 0$
- Eigenspaces: $span\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_m\}$