Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- Vector Space Dimensions
- Infinite Dimensional Vector Spaces
- Relationship between Spanning/Independence/Bases

Back to DEQs!! But higher-order w/linear algebra as a tool!

5.1: 2nd-Order Linear DEQs

Modeling the world with 1st-order DEQs $y' + q(x)y = f(x)$ assumes a simple situation in which the coefficient in front of y'' is zero (and similarly with $y''', y^{(4)}$, etc.).

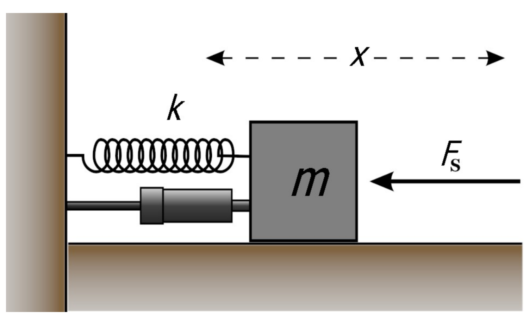
Recall: solving a 1st-order DEQ gives us a 1D family of solutions (e.g., $y = Ce^x$).
But if we assume a more complicated scenario where the coefficient in front of y'' is nonzero, we generate more solutions.
Solving this 2nd-order DEQ (e.g., $y'' + p(x)y' + q(x)y = f(x)$) gives us a 2D family of solutions (e.g., $y = Ae^x + Be^{-x}$).

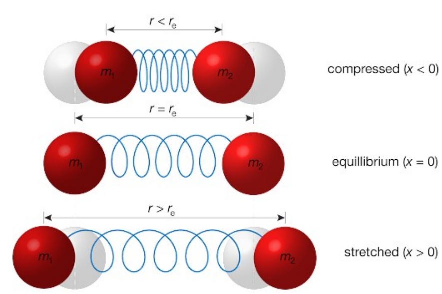
Types of 2nd-order DEQs

- **Linear** (in dep. var): $e^x y'' + \cos(x)y' + (1 + \sqrt{x})y = \tan^{-1}(x)$
- **Non-linear**: $y'' + 3(y')^2 + 4y^3 = 0$
- **Non-homogenous**: $x^2y'' + 2xy' + 3y = \cos x$,

  which is **associated** with **homogenous** DEQ: $x^2y'' + 2xy' + 3y = 0$.

Mechanical Systems





Mass, Spring, Damper (see animation in class)

Many real life scenarios are well modeled by the above mass-spring model: molecular vibration, shock absorbers, atomic bonds, air molecules in sound waves, vibrations in crystals, inductor-capacitor circuits, gravitational/orbital approximations, vibrating strings, tendon/ligament vibrations, etc.

We create this model by combining the spring's force and a damping force together.

Hooke's Law: $F_S = -kx$, where $k > 0$. (Spring Force)

Linear Damping: $F_R = -cv = -c\frac{dx}{dt}$, where $c > 0$. (Resistance/Damping Force)

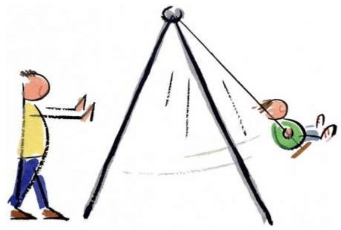
Recall: Position $= x(t)$, Velocity $= v(t) := \frac{dx}{dt}$, Acceleration $= a(t) := \frac{dv}{dt} = \frac{d^2x}{dt}$.

Now we'll use Newton's Second Law of Motion ($F = ma = m\frac{d^2x}{dt^2}$) to create our DEQ:

$F = F_S + F_R = m\frac{d^2x}{dt^2}$, or $mx'' + cx' + kx = 0$. (homogeneous spring model w/damping)

Some models include an external force $F(t)$ that's influencing the natural vibrations.

$mx'' + cx' + kx = F(t)$ (model which includes damping and non-homogeneous external force)



External Periodic Force

More General Mathematical Treatment

Linear 2nd-order DEQs: $A(t)x'' + B(t)x' + C(t)x = F(t)$ or $A(x)y'' + B(x)y' + C(x)y = F(x)$.

Normal Form: $y'' + p(x)y' + q(x)y = f(x)$, obtained if $A(x) \neq 0$ on the interval of interest.

Superposition of Homogeneous DEQ Solutions Thm: If y_1, y_2 are sols to $y'' + p(x)y' + q(x)y = 0$, and c_1, c_2 are constants, then $y = c_1y_1 + c_2y_2$ is also a soln. (not necessarily the general soln) This generalizes to nth-order DEQs!

Proof: We are given that $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and $y_2'' + p(x)y_2' + q(x)y_2 = 0$.

We must show that $y'' + p(x)y' + q(x)y = 0$ when substituting in $y = c_1y_1 + c_2y_2$.

Observe $y' = c_1y_1' + c_2y_2'$ and $y'' = c_1y_1'' + c_2y_2''$.

Substituting in: $y'' + p(x)y' + q(x)y = (c_1y_1'' + c_2y_2'') + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2)$

$= c_1y_1'' + c_1p(x)y_1' + c_1q(x)y_1 + c_2y_2'' + c_2p(x)y_2' + c_2q(x)y_2$

$= c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2)$

$= c_1 \cdot 0 + c_2 \cdot 0 = 0.$ ■

❗ So sols to homogeneous DEQs form a vector space (subset of function space, and closed under addition and scalar mult).

Existence and Uniqueness of Sols to Linear DEQs Thm: Given: $y'' + p_1(x)y' + p_2(x)y = f(x)$,

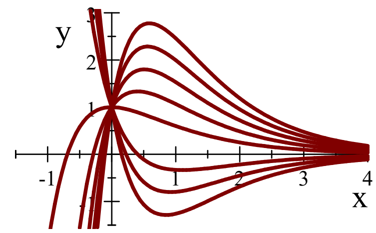
if, near a point $x = a$, the expressions $p_1(x)$, $p_2(x)$, and $f(x)$ are continuous on some interval, then there's a **unique** solution to $y'' + p_1(x)y' + p_2(x)y = f(x)$ on that interval satisfying **initial conditions**: $y(a) = b_0$, $y'(a) = b_1$, for any $b_0, b_1 \in \mathbb{R}$. This generalizes to nth-order DEQs!

Recall that each 1st-order DEQ gave a unique soln for each point $y(a) = b$.

And note that this thm includes *nonhomogeneous* DEQs as well!

For the 2nd-order DEQ above, note that if we choose init-cond $y(a) = b_0$, there's still an infinite number of sols based on our choice of init-cond $y'(a) = b_1$.

Geometrically, this means that for every pt in the plane, we can choose any (finite) slope we want, and there will be a soln going through that pt w/that slope.



$y''+3y'+2y = 0$ w/$y(0) = 1$, different slopes

Recall that in $\mathbb{R}^2$, we needed two linearly independent *vectors* to span the vector space. Similarly, to span the soln set $\mathbb{S}$ of a homogeneous 2nd-order DEQ, you need two linearly independent *functions*, which in our case are y_1, y_2.

And as with the vectors in $\mathbb{R}^2$, y_1, y_2 are **linearly independent** if they are not constant multiples of each other.

$y_1 \neq k y_2$ where $k \in \mathbb{R}$.

However, it's not always clear cut whether two functions are constant multiples of each other.
For instance: $e^x(\sin x + \cos x)$ is a constant multiple of $e^x \sin\left(x + \frac{\pi}{4}\right)$.

Independence of Functions

Given a DEQ $y'' + p_1(x)y' + p_2(x)y = f(x)$, and any two sols y_1, y_2, let's develop the condition under which we can say that the soln $y = c_1 y_1 + c_2 y_2$ represents the general solution (spans the entire solution space).

We need y_1, y_2 to be linearly independent. How can we check this?

Our existence and superposition thms above for DEQ sols suggests that if $y = c_1 y_1 + c_2 y_2$ *IS* our general solution (i.e., y_1, y_2 are independent), we can *uniquely* find any particular solution using any init-conds $y(a) = b_0$, $y'(a) = b_1$.

In other words, we should find just one soln to the system:

$$\begin{aligned} c_1 y_1(a) + c_2 y_2(a) &= b_0 \\ c_1 y_1'(a) + c_2 y_2'(a) &= b_1 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Observe that we can solve for c_1, c_2 uniquely if the determinant of the matrix is nonzero.

However, for y to be the **general soln**, the determinant must be nonzero for e**very choice of** a.
This suggests a method for identifying functions which are linearly independent.

Wronskian (denoted by: W):

Given $y_1(x),\ y_2(x)$ we denote $W(x) = W(y_1,y_2) := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$

This generalizes to n equations giving an $n \times n$ determinant with $n-1$ derivatives.

Wronskian of Sols Thm: If y_1,y_2 are sols to $y'' + p_1(x)y' + p_2(x)y = 0$ on an interval I where p_1,p_2 are continuous, then:

- ♦ y_1,y_2 are dependent *iff* $W(y_1,y_2) = 0$, at each pt x in I.
- ♦ y_1,y_2 are independent *iff* $W(y_1,y_2) \neq 0$ at each pt x in I.

This generalizes to nth-order DEQs with $p_1,\ldots,p_n$, giving n sols $y_1,\ldots,y_n$, and $c_1,\ldots,c_n \in \mathbb{R}$.

So now that we know our sols are independent, we can form a general solution…

General Sols of Homogeneous 2nd-Order DEQs Thm: Let y_1,y_2 be linearly independent solns of $y'' + p(x)y' + q(x)y = 0$, with p,q continuous on some interval I. If Y is any soln whatsoever on I, then there exists $c_1,c_2 \in \mathbb{R}$ such that $Y(x) = c_1y_1 + c_2y_2$, for all x on I.

Proof: Choose $a \in I$. Consider:

$$c_1y_1(a) + c_2y_2(a) = Y(a),$$
$$c_1y_1'(a) + c_2y_2'(a) = Y'(a).$$

$$\begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y(a) \\ Y'(a) \end{bmatrix}$$

Observe: $W(y_1,y_2) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$ (for all $x \in I$, including $x = a$), since we have independence.

So we can reduce the system to solve for c_1,c_2. But does $Y = c_1y_1 + c_2y_2$ for the rest of $x \in I$?

Define $G(x) := c_1y_1(x) + c_2y_2(x)$.
Observe that this solves the DEQ since it is a linear combination of solutions.

Recall that the uniqueness theorem tells us that solns to a DEQ which satisfy initial conditions $y(a) = b_1$ and $y'(a) = b_2$ are unique.

Note that $G(a) = c_1y_1(a) + c_2y_2(a) = Y(a)$.
and $G'(a) = c_1y_1'(a) + c_2y_2'(a) = Y'(a)$.

So, since both G and Y are solns to the DEQ, and satisfy the same init-conds, it must be that $Y(x) \equiv G(x)$, on I.
And we have $Y(x) = c_1y_1 + c_2y_2$, for all x on I. ■

Example: Given the DEQ $y'' - 4y = 0$, verify that the following are sols: $y_1(x) = e^{2x}, y_2(x) = e^{-2x}$.
Then, use a Wronskian to verify that they are linearly independent. Lastly, form the general soln.

Taking derivatives: $y_1' = 2e^{2x}, \quad y_1'' = 4e^{2x}$. Thus: $y'' - 4y = 4e^{2x} - 4e^{2x} = 0$, as desired.

Similarly: $y_2' = -2e^{-2x}, \quad y_2'' = 4e^{-2x}$. Thus: $y'' - 4y = 4e^{-2x} - 4e^{-2x} = 0$, as desired.

$$\text{Wronskian: } W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4 \neq 0. \quad \checkmark$$

General solution: $y = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{-2x}$.

2nd-Order Homogeneous DEQs w/Constant Coefficients

In general, it's difficult/impossible to solve 2nd-order DEQs.

So let's examine the simpler *linear* 2nd-order *homogeneous* DEQs w/*constant coefficients*: $ay'' + by' + cy = 0$.

To solve this, we need a y such that when you take a derivative of it, or several derivatives, the result is a version of itself a constant (i.e., $ay''+by' = -cy$). (what does this remind you of?)

Note if $y := e^{rx}$, then: $y' = (e^{rx})' = re^{rx} = ry$. And $y'' = r^2 y$.

This implies we might find a soln by making this type of substitution, and then solving for r :

$ar^2 y + bry + cy = 0$ OR $(ar^2 + br + c)y = 0$,

$ar^2 + br + c = 0$ (note $y = e^{rx} \neq 0$), which we solve w/the quadratic formula. This leads to the following:

Characteristic Equation Algorithm: To solve $ay'' + by' + cy = 0$, replace y'', y', y with $r^2, r, 1$.

Then, algebraically solve the *characteristic equation* ($ar^2 + br + c = 0$) for r.

♦ If sols r_1, r_2 are real & distinct, $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is the **general soln**, and the soln space has basis $\{e^{r_1 x}, \ e^{r_2 x}\}$.

♦ If $r_1 = r_2$, then $y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$ is the general soln, and the soln space has basis $\{e^{r_1 x}, \ xe^{r_1 x}\}$.

This generalizes to nth-order DEQs with $y^{(n)}, \dots y', y$ and $r^n, \dots r^2, r, 1$.

Example: Find the general solution to: $3y'' - 12y' + 12y = 0$.

Characteristic equation: $3r^2 - 12r + 12r \quad \Rightarrow \quad r^2 - 4y + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0$.

So $r = 2$ is a repeated root, and the general solution is: $y(x) = c_1 e^{2x} + c_2 x e^{2x}$.

Exercises 5.1 

What did we learn?

- DEQs of Mechanical Systems
- Superposition of Homogeneous 2nd-order DEQ Sols
- Existence and Uniqueness for Sols to Linear 2nd-order DEQs
- Wronskians of 2nd-order Sols
- General Sols of Homogeneous 2nd-order DEQs
- Characteristic Equation Algorithm for linear homogeneous w/constant coefficients