Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

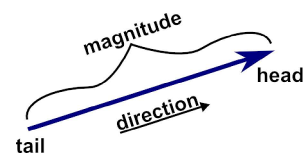
- Calculating Determinants $|\mathbf{A}|$ when $n > 3$
- Calculating Determinants More Easily
- Matrix Transpose $\mathbf{A}^T$
- Cramer's Rule: $\mathbf{A}\vec{x} = \vec{b} \quad \Rightarrow \quad x_i = \frac{|\mathbf{B}_i|}{|\mathbf{A}|}$.
- Inverting Matrices using Cofactors $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T$

4.1: The Vector Space $\mathbb{R}^n$

Review: what are vectors?

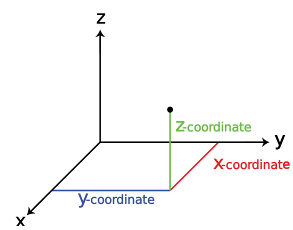
Vectors $\vec{x}$ are ordered sets of numbers (e.g., $(3,1)$).

In $\mathbb{R}^n$, they represent a direction and magnitude $|\vec{x}|$, but in general they don't represent a location!



If you want your vector to represent a position, you can simply refer to it as a *position vector*.

(This tells people you are placing the tail at the origin.)

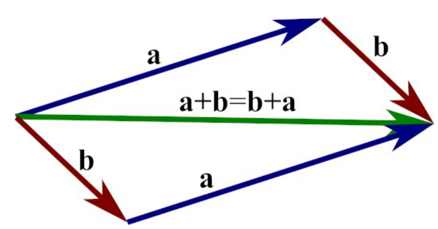


Position vector (x,y,z)

What vector operations are there?

Vector Addition: If $\vec{a} = (1,2,3)$ and $\vec{b} = (4,5,6)$ are vectors, then:

$$\vec{a} + \vec{b} = \big(1+4,\ 2+5,\ 3+6\big) = \big(5,\ 7,\ 9\big).$$

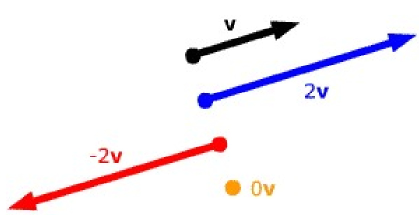


Vector Addition

Scalar Multiplication: multiplying a vector $\vec{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\vec{v} \in V$.

$$c\vec{v} = c\langle v_1, v_2\rangle = \langle cv_1, cv_2\rangle$$

Algebraic



Geometric

Length (magnitude) of Vector $\vec{x} := (x_1, x_2, \ldots, x_n)$: It's a generalization of the Pythagorean theorem:

$$|\vec{x}|^2 = x_1^2 + x_2^2 + \ldots + x_n^2 \quad \text{or} \quad |\vec{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\,.$$

Homogeneity: $\left|c\vec{x}\right| = |c|\left|\vec{x}\right|$.

Proof for $n = 2$: $\left|c\vec{x}\right| = |c(x,y)| = |(cx,cy)|$

$$= \sqrt{c^2x^2 + c^2y^2}$$

$$= |c|\sqrt{x^2+y^2} = |c|\left|\vec{x}\right|. \quad \blacksquare$$

Example (when $c = -3$ and $\vec{a} = (1,2,3)$): Show homogeneity property for: $\left|-3\vec{a}\right|$.

$$\left|-3\vec{a}\right| = |(-3,-6,-9)| = \sqrt{3^2+6^2+9^2} = 3\sqrt{14}, \text{ and } |-3||(1,2,3)| = 3\sqrt{1^2+2^2+3^2} = 3\sqrt{14}.$$

$\mathbb{R}^2$, the set of all 2D vectors is called a **vector space**.

What's a Vector Space?

Below are the properties that a mathematical object must obey to be a vector space:

- ♦ $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, [additive commutivity]
- ♦ $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$, for any $\vec{w}$. [additive associativity]
- ♦ $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$ [additive identity, where $\vec{0} = (0,0,0,\dots)$]
- ♦ $1(\vec{u}) = \vec{u}$ [scalar multiplicative identity]
- ♦ $\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$ [additive inverse, where $-(1,2,3) = (-1,-2,-3)$]
- ♦ $r(\vec{u}+\vec{v}) = r\vec{u}+r\vec{v}$, for any $r \in \mathbb{R}$ [scalar distributivity over vector addition]
- ♦ $(r+s)\vec{u} = r\vec{u}+s\vec{u}$, for any $r,s \in \mathbb{R}$ [vector distributivity over scalar addition]
- ♦ $r(s\vec{u}) = (rs)\vec{u}$ [scalar multiplicative associativity]

Can we prove that $\mathbb{R}^n$ (the set of all n-dim vectors) has the properties of a vector space?

Let's show it for **Scalar Distributivity Over Vector Addition**: $r(\vec{u}+\vec{v}) = r\vec{u}+r\vec{v}$, for any $r \in \mathbb{R}$.

Proof: Given $\vec{u} = (u_1, \dots, u_n)$ and $\vec{v} = (v_1, \dots, v_n)$, then:

$$r(\vec{u}+\vec{v}) = r\left((u_1, \dots, u_n) + (v_1, \dots, v_n)\right)$$

$$= r\left(u_1+v_1, \dots, u_n+v_n\right) = \left(r(u_1+v_1), \dots, r(u_n+v_n)\right)$$

$$= \left(ru_1+rv_1, \dots, ru_n+rv_n\right)$$

$$= \left(ru_1, \dots, ru_n\right) + \left(rv_1, \dots, rv_n\right)$$

$$= r\left(u_1, \dots, u_n\right) + r\left(v_1, \dots, v_n\right) = r\vec{u}+r\vec{v}. \quad \blacksquare$$

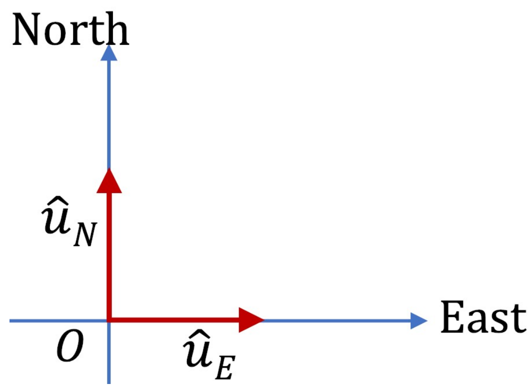
And similarly w/the other properties.

Linearly Dependent/Independent

Roughly speaking, two vectors are considered **independent** if they each give you the ability to move in a different direction.

For instance, if a car was made that could only go east-west, this would be a problem, because it wouldn't be able to access much of the city.

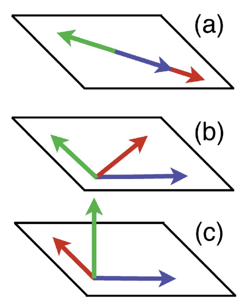
For instance, let $\hat{u}_E = (1,0)$, and $\hat{u}_N = (0,1)$ represent two vectors. And let $\vec{v} := -3 \cdot \hat{u}_E = (-3,0)$ be a third vector.



One way to think about *independence* is that you can't go some distance along $\hat{u}_E$, and then some distance along $\hat{u}_N$, and get back to where you started. In other words, there is no $c_1, c_2 \neq 0$ such that $c_1\hat{u}_E + c_2\hat{u}_N = \vec{0}$.

However, you *can* go some distance along $\hat{u}_E$, and then some distance along $\vec{v}$, and get back to where you started, note that: $3 \cdot \hat{u}_E + \vec{v} = \vec{0}$. This is because they are *dependent*.

More Generally

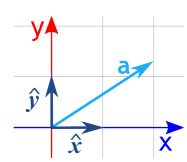


Linearly Dependent Vectors Theorem: Vectors $\vec{u}_1,\ \vec{u}_2, \ldots, \vec{u}_n$ are linearly **dependent** if and only if there exist $a_1, a_2, \ldots, a_n \in \mathbb{R}$, such that $a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = \vec{0}$, where at least one of the a_i is **non-zero**.

Put another way: Vectors $\vec{u}_1,\ \vec{u}_2, \ldots, \vec{u}_n$ are linearly **dependent** if …
$\vec{u}_k = a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_3\vec{u}_n$, for some $1 \leq k \leq n$ and $a_1, a_2, \ldots a_n \in \mathbb{R}$.

Why is the second statement equivalent to the first? Because if one of the a_i is nonzero in the first statement, then you can move the rest of the terms to the other side of the equation, and then divide by the nonzero a_i, resulting in the equation given in the second statement.

Important Consequence: If you have two linearly independent vectors $\vec{u}_1, \vec{u}_2$ in $\mathbb{R}^2$, then any other $\vec{a}$ in $\mathbb{R}^2$ is a linear combination of $\vec{u}_1, \vec{u}_2$. In other words, $\vec{a} = s\vec{u}_1 + t\vec{u}_2$, for some $s, t \in \mathbb{R}$. A similar statement can be made in $\mathbb{R}^3$ for three linearly independent vectors.



$\vec{a} = 2\vec{x} + 1.3$

While it is easy to see that the standard unit vectors are independent, what about other vectors, especially when there are a lot of them in $\mathbb{R}^n$? How can we tell?

Linearly Independent Vectors Theorem: Vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in \mathbb{R}^n$ are linearly **independent**

$$\text{if and only if } (\textit{iff}) \ \left|\vec{u}_1\ \vec{u}_2\ \ldots\ \vec{u}_n\right| := \begin{vmatrix} u_{11} & u_{21} & \ldots & u_{n1} \\ u_{12} & u_{22} & \ldots & u_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n} & u_{2n} & \ldots & u_{nn} \end{vmatrix} \neq 0.$$

Observe we had the requirement that we must have the same number of vectors n as the dimension of our space $\mathbb{R}^n$.
We will learn an alternate method in section 4.3 for when we have fewer vectors.

Proof: Recall that, by definition, $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are linearly **independent** *iff*
$a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = \vec{0}$ implies $a_1 = a_2 = \ldots = a_n = 0$. Written another way:

$$\mathbf{U}\vec{a} := [\vec{u}_1 \ \ \vec{u}_2 \ \ \ldots \ \ \vec{u}_n]\vec{a} = \begin{bmatrix} u_{11} & u_{21} & \ldots & u_{n1} \\ u_{12} & u_{22} & \ldots & u_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n} & u_{2n} & \ldots & u_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ implies only the trivial soln.}$$

But by the corollary to Theorem 7 in section 3.5, the preceding is true *iff* **U** is invertible.

That is, *iff* $|\mathbf{U}| \neq 0$. ■

Bases

Basis for a Vector Space V: A basis is a set of linearly **independent** vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in V such that
every $\vec{v}$ in V can be expressed as a linear combination: $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n$, for some $a_1, a_2, \ldots a_n \in \mathbb{R}$.

A basis is important, because it has the **minimal number of vectors you need to get anywhere you want in the space**.

Particularly for $\mathbb{R}^n$, you need n linearly independent vectors.

For $\mathbb{R}^3$, a convenient (standard) basis is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$,
so that $(a,b,c) = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$, for any $a,b,c \in \mathbb{R}$.

However, $\{(1,2,3),\ (1,5,7),\ (3,0,13)\}$ is also a basis.

Subspaces

Sometimes you are working in a vector space, but you are particularly interested in a subset of your space.

Subspace: A non-empty set of vectors $W := \{\vec{v}_1, \vec{v}_2, \ldots\}$, which are a subset of a vector space V ($W \subseteq V$)

is said to be a subspace if, for all $c \in \mathbb{R}$ and $\vec{v}_i, \vec{v}_j$ in W, we have:

$\vec{v}_i + \vec{v}_j$ is in W, [**closed** under addition]

and $c\vec{v}_i$ is in W. [**closed** under scalar multiplication]

Fact: Subspaces are vector spaces! (but usually smaller) Properties of vector space are "inherited" from parent vector space.

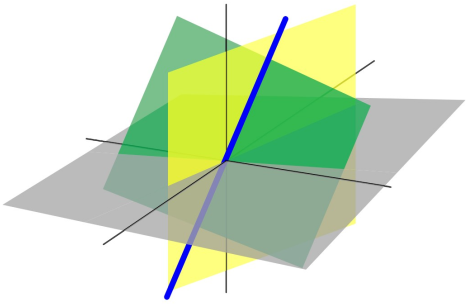
Examples of Subspace:

$\{\vec{0}\}$ in $\mathbb{R}^n$ is a subspace,

Any line through the origin in $\mathbb{R}^n$ with $n \geq 1$,

Any plane through the origin in $\mathbb{R}^n$ with $n \geq 2$,

Any m-hyperplane (?!?) through the origin in $\mathbb{R}^n$ with $n \geq m$.



Vector space and subspaces

Example: Show that V, defined as the set of all (x, y, z) such that $x - 4y + 2z = 0$, is closed under addition and under multiplication by scalars, and is therefore a subspace of $\mathbb{R}^3$.

Let's use the given relation to replace one of the variables. Solving for x: $x = 4y - 2z$.

Replacing x, we have vectors of the form: $(4y - 2z, y, z)$.

Closed under addition? Choosing two such vectors: $(4y_1 - 2z_1, y_1, z_1), (4y_2 - 2z_2, y_2, z_2)$.

Adding them: $(4y_1 - 2z_1, y_1, z_1) + (4y_2 - 2z_2, y_2, z_2) = (4y_1 - 2z_1 + 4y_2 - 2z_2, y_1 + y_2, z_1 + z_2)$

$= (4(y_1 + y_2) - 2(z_1 + z_2), y_1 + y_2, z_1 + z_2)$,

where this vector has the correct form: first component is four times the second component minus two times the third: $(4y - 2z, y, z)$.

Closed under scalar multiplication? $c(4y_1 - 2z_1, y_1, z_1) = (4cy_1 - 2cz_1, cy_1, cz_1)$,

where this vector has the correct form: first component is four times the second component minus two times the third: $(4y - 2z, y, z)$.

Therefore this is a subspace of $\mathbb{R}^3$.

Video Tutorial (visually rich and intuitive): https://youtu.be/fNk_zzaMoSs

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Exercises 4.1

What did we learn?

- What are Vectors
- What's a Vector Space
- When are Vectors Dependent/Independent
- Bases
- Subspaces