Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

---

Previous Lecture

- Matrix Multiplicative Identity
- Matrix Determinant and Inverses
- Matrix Exponents
- Solving Systems w/Matrix Inverses
- Matrix invertibility

3.6: Determinants and Inverses when n $>$ 2

Suppose I give you a physical system (e.g., population) $\mathbf{A}\vec{x}_n = \vec{x}_{n+1}$, where $\vec{x}_n$ represents the current population of various species in the forest, matrix **A** encodes how the variables interact (predation, survival rates, migration, reproduction), and $\vec{x}_{n+1}$ is the population in the following year.

Problem: If you tally the population of these species today $\vec{x}_n$, can you *uniquely* determine what the population was last year $\vec{x}_{n-1}$? Could *different* past populations levels have led to the same present levels?
The determinant is the number that tells us which situation we're in.

The system $\mathbf{A}\vec{x} = \vec{b}$ always determines the future, but determinants tell us **whether the system remembers the past**.

https://youtu.be/Ip3X9LOh2dk?si=XrKN95mOybXnDRDU&t=308

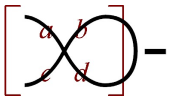
Mathematically, remembering the past means having the ability to calculate an inverse $\mathbf{A}^{-1}$.

Below, we'll also learn how to use determinants to solve large linear systems (Cramer's rule).

Notation/Calculation

The determinant of **A** is a number notated as: $\det \mathbf{A} = |\mathbf{A}| = |a_{ij}| = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Recall the determinant of a 2×2 matrix is calculated as: $|\mathbf{A}| = ad - bc$.



Determiniants of n>2 **Square Matrices**: For these, we need some new tools. e.g, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

We'll make use of a "checkerboard" of signs: $\begin{vmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \vdots \\ \vdots & \vdots & \dots & \ddots \end{vmatrix}$.

Definition: the **minor** (M_{ij}) for a matrix's a_{ij}th entry is found by ignoring the ith row and the jth column,

and taking the determinant of the remaining elements. So, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

Definition: the **cofactor** (A_{ij}) for a matrix's a_{ij}th entry is $A_{ij} :=$ (checkerboard sign)M_{ij}.

$$\text{So, } A_{12} = (-1)M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}.$$

Definition: Finally, the determinant of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is defined as:

$\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$, where A_{ij} is the ijth cofactor.

Example: For a 3×3 matrix, this would be...

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \text{ and similarly for } n \geq 2.$$

$$\text{Or, } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

Notice how easy these calculations would be if some of the a_{ij} were 0.

Later, we'll find out how to arrange for this to happen.

PRO TIP

Above, we "expanded" our determinant along the **top row of the original matrix**.
However, you can calculate the determinant using *any* row or column.
Indeed, if there is a row or column with entries that are zero (or that you can manipulate to be zero), it is strategic to use that row or column to simplify your determinant calculation.

Corollary: If $\mathbf{A}$ contains either a zero column vector or a zero row vector, then $|\mathbf{A}| = 0$.
This follows since we can choose this zero row or column to do our expansion, and the resulting expression will have cofactor coefficients of zero.

Determinant Properties

♦ $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$, or equivalently, $\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}$.

As a direct result of the above property, we can prove $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$. (recall $\mathbf{A}$ invertible implies $|\mathbf{A}| \neq 0$)

Proof: Observe that the result assumes $\mathbf{A}^{-1}$ and $\mathbf{A}$ exist, and therefore we can say:

$\mathbf{AA}^{-1} = \mathbf{I}$. Then, using the determinant property on this equation:

$|\mathbf{AA}^{-1}| = |\mathbf{A}||\mathbf{A}^{-1}| = |\mathbf{I}| = 1$. And dividing by $|\mathbf{A}|$ gives us:

$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$. ■

Recall the Important Property: $\mathbf{A}$ is invertible if $|\mathbf{A}| \neq 0$.

Determinant "Method of Elimination": (calculating determinants more easily)

We can: **Pull constants from rows or columns** (bolded text indicates changes, assume k is some constant.)

$$\begin{vmatrix} \mathbf{ka}_{11} & \mathbf{ka}_{12} & \mathbf{ka}_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \mathbf{k}\begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and}$$

$$\begin{vmatrix} a_{11} & \mathbf{ka}_{12} & a_{13} \\ a_{21} & \mathbf{ka}_{22} & a_{23} \\ a_{31} & \mathbf{ka}_{32} & a_{33} \end{vmatrix} = \mathbf{k}\begin{vmatrix} a_{11} & \mathbf{a}_{12} & a_{13} \\ a_{21} & \mathbf{a}_{22} & a_{23} \\ a_{31} & \mathbf{a}_{32} & a_{33} \end{vmatrix}.$$

Proof for $n = 3$: (done in class).

Switch columns or rows by changing sign.

$$\begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and}$$

$$\begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & a_{13} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & a_{23} \\ \mathbf{a}_{31} & \mathbf{a}_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} \mathbf{a}_{12} & \mathbf{a}_{11} & a_{13} \\ \mathbf{a}_{22} & \mathbf{a}_{21} & a_{23} \\ \mathbf{a}_{32} & \mathbf{a}_{31} & a_{33} \end{vmatrix}.$$

Proof for $n = 3$. (try this one yourself, take the determinant of the matrices in the above equation).

Can add a multiple of a row to another row, and can do the same with columns.

$$\begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ b & c & d \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \mathbf{a}_{11}\mathbf{+kb} & \mathbf{a}_{12}\mathbf{+kc} & \mathbf{a}_{13}\mathbf{+kd} \\ b & c & d \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and}$$

$$\begin{vmatrix} \mathbf{a}_{11} & a_{12} & b \\ \mathbf{a}_{21} & a_{22} & c \\ \mathbf{a}_{31} & a_{32} & d \end{vmatrix} = \begin{vmatrix} \mathbf{a}_{11}\mathbf{+kb} & a_{12} & b \\ \mathbf{a}_{21}\mathbf{+kc} & a_{22} & c \\ \mathbf{a}_{31}\mathbf{+kd} & a_{32} & d \end{vmatrix}.$$

Proof for $n = 3$. (try this one yourself, take the determinant of the matrices in the above equation).

Identical rows or columns means determinant is zero.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b & c & d \\ b & c & d \end{vmatrix} = 0, \text{ and} \begin{vmatrix} a_{11} & a & a \\ a_{21} & b & b \\ a_{31} & c & c \end{vmatrix} = 0.$$

Proof for $n = 3$: (done in class).

Determinant of (upper or lower) triangular matrices is the product of the main diagonal.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33}.$$

Proof for $n = 3$: (done in class).

Therefore $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = 0.$

Example: Use the method of elimination (*determinant* row/column operations) to evaluate: $\begin{vmatrix} -2 & 5 & 4 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix}$.

$$\Rightarrow \quad c_2 + (-4c_1) \quad \Rightarrow \quad \begin{vmatrix} -2 & 13 & 4 \\ 5 & -17 & 1 \\ 1 & 0 & 5 \end{vmatrix} \quad \Rightarrow \quad c_3 + (-5c_1) \quad \Rightarrow \quad \begin{vmatrix} -2 & 13 & 14 \\ 5 & -17 & -24 \\ 1 & 0 & 0 \end{vmatrix}$$

(note that the above operations can only be performed on determinants, not matrices!)

$$\Rightarrow \quad -\begin{vmatrix} -2 & 13 & 14 \\ 1 & 0 & 0 \\ 5 & -17 & -24 \end{vmatrix} \quad \Rightarrow \quad +\begin{vmatrix} 1 & 0 & 0 \\ -2 & 13 & 14 \\ 5 & -17 & -24 \end{vmatrix} = +1\begin{vmatrix} 13 & 14 \\ -17 & -24 \end{vmatrix} = -74. \quad \text{(Checkerboard and Fish)}$$

(don't need these steps here, but thought I'd show row-swapping/sign-changes)

In addition to the $\left[\mathbf{A} \mid \mathbf{I}\right] \rightarrow \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$ method, there's another method to find inverses when $n > 2$, but we need another tool: *transposes*.

Matrix Transpose

If $\mathbf{A} = \begin{bmatrix} \mathbf{c} & \mathbf{a} & \mathbf{t} \\ p & e & n \\ \mathbf{m} & \mathbf{o} & \mathbf{m} \end{bmatrix}$, then $\mathbf{A}^T = \begin{bmatrix} \mathbf{c} & p & \mathbf{m} \\ \mathbf{a} & e & \mathbf{o} \\ \mathbf{t} & n & \mathbf{m} \end{bmatrix}$.

Transpose Properties

- $(\mathbf{A}^T)^T = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
- $(c\mathbf{A})^T = c\mathbf{A}^T$
- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ (recall: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$)
- $\det(\mathbf{A}^T) = \det \mathbf{A}$

Cramer's Rule

It's a way to easily solve a system $\mathbf{A}\vec{x} = \vec{b}$ using determinants.

$$\text{Given: } \mathbf{A}^{3\times3} := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ with } |\mathbf{A}| \neq 0, \text{ and given } \vec{b} := \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

$$\text{Define: } |\mathbf{B}_1| := \begin{vmatrix} \mathbf{b}_1 & a_{12} & a_{13} \\ \mathbf{b}_2 & a_{22} & a_{23} \\ \mathbf{b}_3 & a_{32} & a_{33} \end{vmatrix}, \quad |\mathbf{B}_2| := \begin{vmatrix} a_{11} & \mathbf{b}_1 & a_{13} \\ a_{21} & \mathbf{b}_2 & a_{23} \\ a_{31} & \mathbf{b}_3 & a_{33} \end{vmatrix}, \quad |\mathbf{B}_3| := \begin{vmatrix} a_{11} & a_{12} & \mathbf{b}_1 \\ a_{21} & a_{22} & \mathbf{b}_2 \\ a_{31} & a_{32} & \mathbf{b}_3 \end{vmatrix}.$$

The solution to $\mathbf{A}\vec{x} = \vec{b}$ is $\vec{x} = \langle x_1, x_2, x_3 \rangle$, where:

$$x_1 = \frac{|\mathbf{B}_1|}{|\mathbf{A}|}, \qquad x_2 = \frac{|\mathbf{B}_2|}{|\mathbf{A}|}, \qquad x_3 = \frac{|\mathbf{B}_3|}{|\mathbf{A}|}, \text{ and similarly with matrices sized } n > 3.$$

Example: Use Cramer's rule to solve the system:

$$\begin{aligned} 2x_1 - 5x_3 &= -3 \\ 4x_1 + 3x_3 - 5x_2 &= 3 \\ -2x_1 + x_2 + x_3 &= 1 \end{aligned}$$

$$\mathbf{A}\vec{x} = \vec{b} \qquad |\mathbf{A}| = \begin{vmatrix} 2 & 0 & -5 \\ 4 & -5 & 3 \\ -2 & 1 & 1 \end{vmatrix} \qquad \text{(let's further simplify the column with the zero in it)}$$

$$|\mathbf{A}| = \begin{vmatrix} 2 & 0 & -5 \\ -6 & 0 & 8 \\ -2 & 1 & 1 \end{vmatrix} = -1(16 - 30) = 14 \neq 0. \qquad \checkmark$$

$$x_1 = \frac{|\mathbf{B}_1|}{|\mathbf{A}|} = \frac{1}{14}\begin{vmatrix} \mathbf{-3} & 0 & -5 \\ \mathbf{3} & -5 & 3 \\ \mathbf{1} & 1 & 1 \end{vmatrix} = -\frac{8}{7}, \quad x_2 = \frac{|\mathbf{B}_2|}{|\mathbf{A}|} = \frac{1}{14}\begin{vmatrix} 2 & \mathbf{-3} & -5 \\ 4 & \mathbf{3} & 3 \\ -2 & \mathbf{1} & 1 \end{vmatrix} = -\frac{10}{7}, \qquad x_3 = \frac{1}{14}\begin{vmatrix} 2 & 0 & \mathbf{-3} \\ 4 & -5 & \mathbf{3} \\ -2 & 1 & \mathbf{1} \end{vmatrix} = \frac{1}{7}.$$

$$\text{So, } \vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \frac{1}{7}\begin{bmatrix} -8 & -10 & 1 \end{bmatrix}^T.$$

Matrix Inverse $\mathbf{A}^{-1}$ when $\mathbf{n} > 3$

In addition to the $\left[\mathbf{A} \mid \mathbf{I}\right] \to \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$ method, we have the following.

Recall we learned about cofactors A_{mn}, which are signed sub-determinants of $\mathbf{A}$.
Let's create a new matrix $[A_{mn}]$, which consists of these cofactors (determinants), called the *cofactor* matrix of $\mathbf{A}$.
It turns out that if you multiply $\mathbf{A}$ by the transpose of the cofactor matrix $[A_{mn}]$, you get: $\mathbf{A}[A_{mn}]^T = |\mathbf{A}|\mathbf{I}$.

This is amazing, as we now have access to an inverse.
That is, all we must do is divide by $|\mathbf{A}|$: $\mathbf{A}\frac{1}{|A|}[A_{mn}]^T = \mathbf{I}$. In other words, $\frac{1}{|A|}[A_{mn}]^T$ is the inverse of $\mathbf{A}^{-1}$.
This also suggests the inverse doesn't exist if $|\mathbf{A}| = 0$ (that is, if the transformation loses dimensions).

Cofactor Matrix Inverse Method Theorem: If $\mathbf{A}$ is invertible (so first check the determinant), we can construct a **cofactor** matrix $[A_{mn}]$, which consists of the **cofactors** of $\mathbf{A}$ (*NOT* $\mathbf{A}$'s components). And then: $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T$.

Proof: We need to solve $\mathbf{AX} = \mathbf{I}$ for $\mathbf{X}$. So, for each column vector $\vec{x}_i$ in $\mathbf{X}$, use Cramer's rule to solve $\mathbf{A}\vec{x}_i = \vec{e}_i$. ■

Row reduction $\left[\mathbf{A} \mid \mathbf{I}\right] \to \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$ is our calculator. It's how we *usually* compute inverses.

The cofactor (determinant) method is our microscope. It reveals the structure behind invertibility.

It tells us that inverses exist because determinants encode a matrix's geometry (how it transforms the vector space).

$$\text{If } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[A_{mn}]^T = \frac{1}{\det \mathbf{A}}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^{\mathbf{T}} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

... where A_{mn} are the *cofactors* (NOT the components) of $\mathbf{A}$.

Example: Use the cofactor method to find the inverse ($\mathbf{A}^{-1}$) of: $\mathbf{A} = \begin{bmatrix} 2 & 4 & -3 \\ 2 & -3 & -1 \\ -5 & 0 & -3 \end{bmatrix}$.

Then, use it to solve $\mathbf{A}\vec{x} = \vec{b}$ where $\vec{b} = [0,1,1]^T$. (HINT $\det \mathbf{A} = 107$)

Recall: $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T$. **Cofactors**: $A_{mn} = (\text{Checkerboard})(\text{Fish})$.

So, $A_{11} = (+)(9-0) = 9$, $A_{12} = (-)(-6-5) = 11$, $A_{13} = (+)(0-15)\ldots$ So, $[A_{mn}] = \begin{bmatrix} 9 & 11 & -15 \\ 12 & -21 & -20 \\ -13 & -4 & -14 \end{bmatrix}$.

And the transpose: $[A_{mn}]^T$ is $\begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}$, so $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T = \frac{1}{107}\begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}$.

Solution: $\vec{x} = \frac{1}{107}\begin{bmatrix} 9 & 12 & -13 \\ 11 & -21 & -4 \\ -15 & -20 & -14 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{107}\begin{bmatrix} -1 \\ -25 \\ -34 \end{bmatrix}$.

Video Tutorials (visually rich and intuitive): https://youtu.be/uQhTuRlWMxw https://youtu.be/Ip3X9LOh2dk

Exercises 3.6 💪

What did we learn?

- Calculating Determinants $|\mathbf{A}|$ when $n > 3$
- Calculating Determinants More Easily
- Matrix Transpose $\mathbf{A}^T$
- Cramer's Rule: $\mathbf{A}\vec{x} = \vec{b} \quad \Rightarrow \quad x_i = \frac{|\mathbf{B}_i|}{|\mathbf{A}|}$.
- Inverting Matrices using Cofactors $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[A_{mn}]^T$

Prepared by Dr. Jodin Morey. Materials for Other Courses Found at **MathTalker.org**