Previous Lecture

- Matrix Operations
- Matrix Algebraic Properties

3.5: Inverses of Matrices

What if you're attempting to solve a linear system, can you do it after putting it in matrix form: $\mathbf{A}\vec{x} = \vec{b}$?

Wouldn't it be nice if there were an "inverse" $\mathbf{A}^{-1}$, such that you could multiply both sides of the eqn to get:
$\mathbf{A}^{-1}\mathbf{A}\vec{x} = \mathbf{A}^{-1}\vec{b}$ or $\vec{x} = \mathbf{A}^{-1}\vec{b}$? Sometimes this is possible!

Matrix Multiplicative Identity

First, let's define the idea of a multiplicative identity for matrices.

Theory: For each $\mathbf{A}^{m\times n}$, the identity matrix $\mathbf{I}^{n\times n}$ behaves like the multiplicative identity 1 in regular algebra (i.e., $a \cdot 1 = a$).

Proof: To show this, we first obtain two other results.

Fact 1: We can notate $\mathbf{A}$ as $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$. (where we've notated the columns $\vec{a}_i$ as vectors)

Then, for any n-vector $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$, we have: $\mathbf{A}\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$.

Let's try it for a small matrix: $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}\vec{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1a_{11} + x_2a_{12} \\ x_1a_{21} + x_2a_{22} \end{bmatrix}$

$= \begin{bmatrix} x_1a_{11} \\ x_1a_{21} \end{bmatrix} + \begin{bmatrix} x_2a_{12} \\ x_2a_{22} \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2$, and similarly for $n > 2$.

Fact 2: for non-square matrices, $\mathbf{A}^{m\times n}$ and $\mathbf{B}^{n\times p} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_p \end{bmatrix}$,

we have: $\mathbf{AB} = \begin{bmatrix} \mathbf{A}\vec{b}_1 & \mathbf{A}\vec{b}_2 & \dots & \mathbf{A}\vec{b}_p \end{bmatrix}$.

Let's try it for small matrices: $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$.

Observe that: $\mathbf{A}\vec{b}_1 = a_{11}b_{11} + a_{12}b_{21}$, $\mathbf{A}\vec{b}_2 = a_{11}b_{12} + a_{12}b_{22}$, and $\mathbf{A}\vec{b}_3 = a_{11}b_{13} + a_{12}b_{23}$.

And, $\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}\vec{b}_1 & \mathbf{A}\vec{b}_2 & \mathbf{A}\vec{b}_3 \end{bmatrix}, \text{ and similarly for larger matrices.}$$

To show $\mathbf{AI} = \mathbf{A}$, note that $\mathbf{I} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \ldots & \vec{e}_n \end{bmatrix}$, where the $\vec{e}_j$ are the j^{th} standard unit vector:

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ entry.}$$

Now if $\mathbf{A} := \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix}$, then by **Fact 1**: $\mathbf{A}\vec{e}_j = 0 \cdot \vec{a}_1 + \ldots + 1 \cdot \vec{a}_j + \ldots + 0 \cdot \vec{a}_n = \vec{a}_j$.

Therefore, from **Fact 2**: $\mathbf{AI} = \mathbf{A}\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \ldots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}\vec{e}_1 & \mathbf{A}\vec{e}_2 & \ldots & \mathbf{A}\vec{e}_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} = \mathbf{A}$. ■

Matrix Determinants and Inverses

Recall from previous mathematics that if $a \neq 0$, then there is a (unique) number $a^{-1} := \frac{1}{a}$ such that $aa^{-1} = a^{-1}a = 1$.

We refer to a^{-1} as a's (unique) **inverse**, and we say a is **invertible**.

Does such a property exist for matrices? Kind of...

We say $\mathbf{A}$ is **invertible** (also called **non-singular**) if there exists a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$.

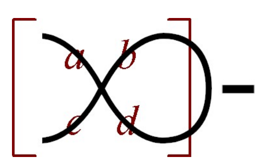
If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}^{-1}$ is $\mathbf{A}'s$ (unique) **inverse**. Only some square matrices: $\mathbf{A}^{n \times n}$ are invertible.

But instead of requiring "$\mathbf{A} \neq 0$", we need something called the **determinant** of $\mathbf{A}$ to be nonzero for this inverse to exist.

2×2 Matrices

Given: $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$\mathbf{A}$ is **invertible** if $ad - bc$ (its **determinant**) is nonzero.



determinant

In which case (in this 2×2 example) we have the **inverse**: $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

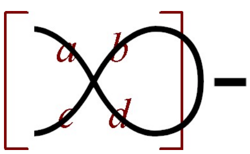
One-over-Determinant, **Interchange**, then **Negate** (**ODIN**)

Proof: $\mathbf{AA}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}.$$

Similarly for $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. ■

Example: Find the inverse (if it exists) for $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Recall: 

and $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. 

determinant

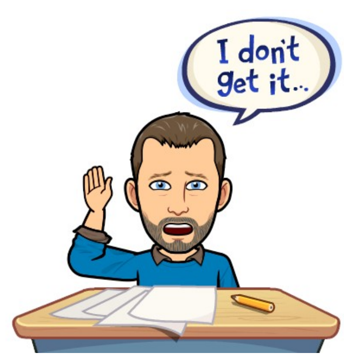
ODIN

So, determinant is: $ad - bc = 1 \cdot 4 - 2 \cdot 3 = -2$. And: $\mathbf{A}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

Example: Find the inverse (if it exists) for $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$.

The determinant is: $ad - bc = 2 \cdot (-3) - 6 \cdot (-1) = 0$. So the inverse doesn't exist.

Why are We Doing This?



Why do we study inverses and determinants so much, since they rely on square matrices. Isn't it a rare thing that the number of variables (columns) should match up w/the number of equations (rows) in what we are studying?

Yes. But in practice, there are methods by which we can convert these problems into ones that involve square matrices. Also, we can look at square parts of non-square matrices, calculate their determinants, and use this information to tell us things about our system. These more sophisticated methods can be studied once we know the basics.

$\mathbf{A}^{-1}$ for Larger Matrices

Convert $[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$ into…

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & 1 & 0 & 0 & a'_{21} & a'_{22} & \cdots & a'_{2n} \\ 0 & 0 & 1 & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{array}\right] = [\mathbf{I} \mid \mathbf{A}^{-1}],$$ using elementary row operations.

Example: Find the inverse (if it exists) for $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Recall: convert $[\mathbf{A} \mid \mathbf{I}]$ into $[\mathbf{I} \mid \mathbf{A}^{-1}]$. So, $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\text{Swap } R_1 \text{ and } R_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\overset{\text{Add } -2R_1 \text{ to } R_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\text{Add } R_2 \text{ to } R_1 \text{ and } R_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & -2 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

$$\overset{\text{Multiply } R_2 \text{ and } R_3 \text{ by } -1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \overset{\text{Add } R_3 \text{ to } R_1 \text{ and } -2R_3 \text{ to } R_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] = [\mathbf{I} \mid \mathbf{A}^{-1}].$$

Let's check our work, does $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$? $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$ ✓

Example: Find the inverse (if it exists) for $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}$.

So, $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 6 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\text{Add } -2R_1 \text{ to } R_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

Here (due to the zero row) we can tell it can never reduce to $[\mathbf{I} \mid \mathbf{A}^{-1}]$, so apparently there's no inverse. 

For fun, let's continue: $\overset{\text{Swap } R_2 \text{ to } R_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & 0 \end{array}\right] \overset{\text{Add } -2R_2 \text{ to } R_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & 0 \end{array}\right]$

Now we're stuck!

Matrix Exponents

An application of matrices and DEQs (which we will learn in a later chapter) is that we can predict the populations of animals species by applying a matirx $\mathbf{A}$ to the current population vector $\vec{x}$, (and other similar applications).

To predict five years in the future, this would require us to calculate $\mathbf{AAAAA}\vec{x} = \mathbf{A}^5\vec{x}$.

Notation: For positive integer n, we define $\mathbf{A}^0 := \mathbf{I}$, $\quad \mathbf{A}^1 := \mathbf{A}$. (same as prior math)

Also: $\mathbf{A}^{n+1} := \mathbf{A}^n\mathbf{A}$, and $\mathbf{A}^{-n} := (\mathbf{A}^{-1})^n$.

The **laws of exponents** are also consistent with prior experience: $\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^{r+s}$, and $(\mathbf{A}^r)^s = \mathbf{A}^{rs}$.

Inverse Manipulations: If $\mathbf{A}$ is invertible, then…

♦ $\mathbf{A}^{-1}$ (the inverse of $\mathbf{A}$) is also invertible, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

Proof: Since $\mathbf{A}$ is invertible, we have: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$.

For $\mathbf{A}^{-1}$ to be invertible, we would need: $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Note this equivalent to the previous line. ■

♦ If n is a positive integer, then $\mathbf{A}^n = \mathbf{AA}\ldots\mathbf{A}$ (multiply n times) is also invertible, and $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$.

Proof: (in class)

If $\mathbf{A}, \mathbf{B}$ are of equal size and invertible, then…

♦ The product $\mathbf{AB}$ is also invertible, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (observe how they switched positions!)

Proof: (in class)

Solving Square Systems w/Matrix Inverses

Theorem: If $\mathbf{A}^{n \times n}$ is invertible, then for the n-vector $\vec{b}$, it's true that $\mathbf{A}\vec{x} = \vec{b}$ has a *unique* solution: $\vec{x} = \mathbf{A}^{-1}\vec{b}$.

Proof: First we show $\vec{x} = \mathbf{A}^{-1}\vec{b}$ *is* a solution.

Substituting it into the LHS of the eqn: $\mathbf{A}\left(\mathbf{A}^{-1}\vec{b}\right) = (\mathbf{AA}^{-1})\vec{b} = \mathbf{I}\vec{b} = \vec{b}$, so $\vec{x}$ **is** a soln.

Next, we must show it is the **only** soln. Here we'll use *proof by contradiction*.

Let's assume the opposite of what we believe is true, that is, that there *is* another *distinct* solution $\vec{x}_1$.

So we have: $\mathbf{A}\vec{x}_1 = \vec{b}$.

Multiplying both sides of this eqn by $\mathbf{A}^{-1}$, we have: $\mathbf{A}^{-1}\mathbf{A}\vec{x}_1 = \mathbf{A}^{-1}\vec{b} \quad \Rightarrow \quad \mathbf{I}\vec{x}_1 = \mathbf{A}^{-1}\vec{b}$

$\Rightarrow \quad \vec{x}_1 = \vec{x} \qquad$ (since $\vec{x} = \mathbf{A}^{-1}\vec{b}$).

But this is a contradiction of our assumption that $\vec{x}_1$ was *distinct* from $\vec{x}$.

Therefore, that assumption must have been incorrect, and in fact there is only the one solution, namely $\vec{x}$. ■

Invertible Square Matrices and Row Operations Theorem (Thm 7): The $n \times n$ matrix **A** is invertible *if and only if* it is row equivalent to the identity matrix **I**. (proof in book)

Corollary: **A** is invertible *if and only if* $\mathbf{A}\vec{x} = \vec{0}$ has only the trivial soln. (trivial soln is: $x_1 = x_2 = \ldots = x_n = 0$).

Properties of Non-Singular Square Matrices $|\mathbf{A}^{n\times n}| \neq 0$

Theorem: The following are equivalent (TFAE):

- **A** is invertible
- **A** is row equivalent to the identity matrix **I**. (Thm 7)
- $\mathbf{A}\vec{x} = \mathbf{0}$ has only the trivial soln. (Thm 7)
- For every n-vector $\vec{b}$, the system $\mathbf{A}\vec{x} = \vec{b}$ has a unique soln. (see proof above)
- For every n-vector $\vec{b}$, the system $\mathbf{A}\vec{x} = \vec{b}$ is consistent.

Video Tutorial (visually rich and intuitive): https://youtu.be/kYB8IZa5AuE

Exercises 3.5 💪

What did we learn?

- Matrix Multiplicative Identity
- Matrix Determinant and Inverses
- Matrix Exponents
- Solving Systems w/Matrix Inverses
- Matrix invertibility