Previous Lecture

- Reduced Row Echelon Form - RREF
- Homogeneous Systems and their sols: $[\mathbf{A}\,|\,\vec{0}]$
- Non-Homogeneous Systems and their sols: $[\mathbf{A}\,|\,\vec{b}]$
- Square Coefficient Systems and their sols: $[\mathbf{A}^{n\times n}\,|\,\vec{0}]$

3.4: Matrix Operations

So, putting linear systems into a matrix gives a convenient way to solve many eqs simultaneously.

It turns out that after assigning matrices some notation, like $\mathbf{A}, \mathbf{B}$, etc., we can think of them as mathematical objects, complete w/their own sense of addition, subtraction, multiplication, etc.

And if we do this, our power to solve complicated eqs (and later DEQs) will be enhanced.

First, let's define what it means for a matrix $\mathbf{A}$ to be equal to a matrix $\mathbf{B}$: *they're the same size, and their elements are equal*.

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \neq \begin{bmatrix} 1 & 9 \\ 4 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The operations below work for matrices of the same size.
If the matrices are not of the same size, then some of the operations won't work (particularly addition).
Matrix multiplication has more flexibility wrt size requirements, which we'll cover below.

Addition: $\mathbf{A}+\mathbf{B} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}.$

Scalar Multiplication: $k\mathbf{A} := k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}.$

Negative of a matrix: $-\mathbf{A} := -1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.$

Therefore, subtraction is a kind of addition: $\mathbf{A}-\mathbf{B} = \mathbf{A} + (-\mathbf{B})$.

Matrix Multiplication

Matrix multiplication is less intuitive. We wish to define it such that systems like:

$$\begin{aligned} 1x + 5y + 8z &= 0 \\ 2x + 6y + 9z &= 2 \\ 3x + 7y + 11z &= 4 \end{aligned},$$

with the coefficient matrix $\mathbf{A} := \begin{bmatrix} 1 & 5 & 8 \\ 2 & 6 & 9 \\ 3 & 7 & 11 \end{bmatrix}$, variable vector $\vec{x} := \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

and constant vector $\vec{b} := \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$, can be written as $\mathbf{A}\vec{x} = \vec{b}$. (where we're multiplying $\mathbf{A}$ and $\vec{x}$)

Thus, matrix multiplication is defined for $\mathbf{A}^{m \times n}$ and $\mathbf{B}^{n \times p}$, where the number of columns (n) in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$. The result is $\mathbf{A}^{m \times n}\mathbf{B}^{n \times p} = \mathbf{C}^{m \times p}$. So the inner indices "cancel."

$$\mathbf{A}^{3 \times 3}\mathbf{x}^{3 \times 1} := \begin{bmatrix} 1 & 5 & 8 \\ 2 & 6 & 9 \\ 3 & 7 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 5y + 8z \\ 2x + 6y + 9z \\ 3x + 7y + 11z \end{bmatrix} = \mathbf{C}^{3 \times 1}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} =$$

See animation in class

And similarly: $\mathbf{a}^{1 \times 3}\mathbf{c}^{3 \times 1} := \begin{bmatrix} 1 & 5 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 1 \cdot 1 + 5(-2) + 8 \cdot 3 = 15 := \mathbf{c}^{1 \times 1}$ (dot product of vectors), and

$$\mathbf{AB} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

When the size of the matrices are not compatible (as defined above), matrix multiplication is not defined.

Algebraic Properties

Now that we have defined operations for matrices, it's easy to discover that the following properties are true.
These are mostly the same properties we learned w/regular numbers.

First, let's define a couple of essential objects: *Identity and Zero Matrices*

$$\mathbf{I} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{0} := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \qquad \text{Note: } \mathbf{AI} = \mathbf{A}, \qquad \mathbf{A} + \mathbf{0} = \mathbf{A}.$$

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ **Additive Commutative Property**

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ **Additive Associative Property**

$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ **Multiplicative Associative Property**

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$, and **Distributive Properties**
$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

You can convince yourself that these are true by coming up with a few simple matrices, and trying them out.

However, unlike regular algebra, AB is usually NOT equal to BA.
(i.e., matrices usually don't have "multiplicative commutivity")

If $\mathbf{A} := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, then notice that

$\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, however $\mathbf{BA} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. So, $\mathbf{AB} \neq \mathbf{BA}$.

Column $\begin{bmatrix} a \\ b \end{bmatrix}$ and row $\begin{bmatrix} e & f \end{bmatrix}$ vector notation

$\begin{bmatrix} a \\ b \end{bmatrix} = (a,b) \neq \begin{bmatrix} a & b \end{bmatrix}$. The parenthetical notation for $\begin{bmatrix} a & b \end{bmatrix}$ is $(a,b)^T$. (the *T*ranspose of (a,b))

Video Tutorial (visually rich and intuitive, although you may have to pause and rewatch parts till it all sinks in): https://youtu.be/XkY2DOUCWMU

Exercises 3.4

What did we learn?

- Matrix Operations
- Matrix Algebraic Properties