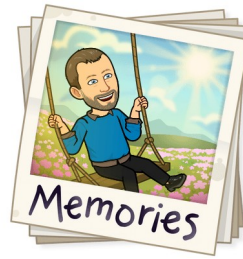


Differential Eqns and Linear Algebra

Textbook: *Differential Equations and Linear Algebra* by Edward and Penney

Previous Lecture

- ◆ Matrices and augmented matrices
- ◆ Elemental row operations
- ◆ Echelon matrix: leading entry/variables, free variables
- ◆ Row equivalence
- ◆ Solving through back substitution



3.3: Reduced Row Echelon Form - RREF

Depending upon how a person does their elementary row operations, they'll end up w/different echelon forms.

This is a problem since in mathematics/science, we want consistent results.

However, *Gauss-Jordan Elimination* is a process by which everyone gets the same result: *Reduced Row Echelon Form - RREF*.

This form also often provides us w/an immediate soln, without the need to do back substitution.

Our task, try to convert:

$$[\mathbf{A} | \vec{b}] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ a_{41} & a_{42} & a_{43} & b_4 \end{array} \right] \text{ into } \left[\begin{array}{ccc|c} 1 & 0 & 0 & b'_1 \\ 0 & 1 & 0 & b'_2 \\ 0 & 0 & 1 & b'_3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{OR into } \left[\begin{array}{ccc|c} 1 & a'_{12} & 0 & b'_1 \\ 0 & 0 & 1 & b'_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

"Reduced Row Echelon Form - RREF"

Unlike non-reduced echelon form, it's unique!!

Continuing our Example:

$$\left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 2 & 6 & 9 & 2 \\ 3 & 7 & 11 & 4 \\ 6 & 14 & 22 & 8 \end{array} \right]$$

$$\begin{array}{l} \text{Add } -2R_1 \text{ to } R_2 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 0 & -4 & -7 & 2 \\ 3 & 7 & 11 & 4 \\ 6 & 14 & 22 & 8 \end{array} \right]$$

Since we already had a 1 in the upper left, we can begin eliminating the other entries in this column.

$$\begin{array}{l} \text{Add } -3R_1 \text{ to } R_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 0 & -4 & -7 & 2 \\ 0 & -8 & -13 & 4 \\ 6 & 14 & 22 & 8 \end{array} \right] \quad \begin{array}{l} \text{Add } -6R_1 \text{ to } R_4 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 0 & -4 & -7 & 2 \\ 0 & -8 & -13 & 4 \\ 0 & -16 & -26 & 8 \end{array} \right]$$

$$\begin{array}{l} \text{Add } -2R_3 \text{ to } R_4 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 0 & -4 & -7 & 2 \\ 0 & -8 & -13 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \text{Add } -2R_2 \text{ to } R_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 8 & 0 \\ 0 & -4 & -7 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{l} \text{Add } 7R_3 \text{ to } R_2 \text{ and } -8R_3 \text{ to } R_1 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 0 & 0 \\ 0 & -4 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} -\frac{1}{4}R_2 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \text{Add } -5R_2 \text{ to } R_1 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Homogeneous Matrices, Homogeneous Linear Systems

Homogeneous linear systems (and their augmented matrices) are easier to solve, and can often serve as building blocks to solve non-homogeneous systems.

Definition: A linear system (like \mathbf{A} above) is **homogeneous** if the constants b_1, b_2, \dots, b_n on the RHS are all zeros.

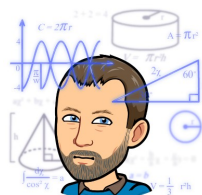
Definition: A soln to a homogeneous linear system (like \mathbf{A} above) is **trivial** if the soln is: $x_1 = x_2 = \dots = x_n = 0$.

You should convince yourself that the trivial soln is always a soln to a homogeneous system.

$$\mathbf{H} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{array} \right] \Rightarrow \begin{array}{l} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \\ a_{41}x + a_{42}y + a_{43}z = 0 \end{array} \Rightarrow \begin{array}{l} a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 0 = 0 \\ a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 0 = 0 \\ a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 0 = 0 \\ a_{41} \cdot 0 + a_{42} \cdot 0 + a_{43} \cdot 0 = 0 \end{array}$$

Observe this is true irrespective of what the a_{ij} are.

What if you knew that \mathbf{H} had at least one nontrivial soln?
Could you conclude something about how many sols it has?



How many sols for a homogeneous linear system \mathbf{H} w/more variables (columns) than equations (rows)?

$$\mathbf{H} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{array} \right]$$

$$\rightarrow \tilde{\mathbf{H}} = \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & 0 \\ 0 & 1 & c_{23} & 0 \end{array} \right]$$



Non-Homogeneous Matrices

$$\mathbf{N} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ a_{41} & a_{42} & a_{43} & b_4 \end{array} \right], \text{ where at least one of the } b_i \neq 0.$$

Observe that after Gauss-Jordan elimination, \mathbf{N} *could* become:

$$\tilde{\mathbf{N}} = \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & d_1 \\ 0 & 1 & c_{23} & d_2 \\ 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 \end{array} \right], \text{ where } d_3 \neq 0.$$

The third row reads: $0 \cdot x + 0 \cdot y + 0 \cdot z = d_3 \neq 0$. But this is impossible! How'd it happen?

We started w/the *assumption* that the system had a soln, and then performed row manipulations in an attempt to find it.

We ended up w/the *absurd* statement that $0 = d_3 \neq 0$. So this tells us that we started with an *absurd assumption*.

In other words, our assumption that there was a soln was apparently absurd.

Moral of the story: if transforming your system to RREF results in a row which reads $0 = c \neq 0$, there are **no sols for the system**.

Observe this couldn't happen w/a homogeneous system, because the reduction process involves multiplying the zeros in the last column by constants, and then adding or subtracting them. But this still leaves these components in the last column as zeros!

But what about a non-homogeneous linear system \mathbf{N} w/more variables (columns) than eqs (rows) how many sols might it have?

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & c_{13} & d_1 \\ 0 & 1 & c_{23} & d_2 \end{array} \right] ??$$

$$\text{OR} \rightarrow \left[\begin{array}{ccc|c} 1 & c_{12} & c_{13} & d_1 \\ 0 & 0 & 0 & d_2 \end{array} \right] ??$$

$n \times n$ Square Coefficient Matrices: \mathbf{A}

What if you had a homogeneous system \mathbf{H} w/the same number of variables (columns) as equations (rows)?

$$\mathbf{H} = [\mathbf{A} | \vec{0}] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array} \right].$$

\mathbf{A} is referred to as a 3×3 square *coefficient matrix*. More generally, we will see $n \times n$ square coefficient matrices.

We'll be interested in discovering when such a system has only the trivial soln $x_1 = \dots = x_n = 0$.

(Why? Because matrices like these can be used to easily solve *non-homogeneous* eqs)

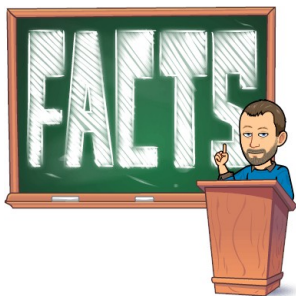
If \mathbf{H} has only the trivial soln, then it must be row equivalent to:

$$\mathbf{H} \stackrel{\text{row}}{\equiv} [\mathbf{I} | \vec{0}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right], \text{ where } \mathbf{I} \text{ is the } 3 \times 3 \text{ identity matrix.}$$

That is, \mathbf{I} is a matrix with 1s on its **principal diagonal** (the one from upper left to lower right) and zeros elsewhere.

More generally:

Square Homogeneous System Thm: Let \mathbf{A} be an $n \times n$ matrix. Then the homogeneous system w/coefficient matrix \mathbf{A} has only the trivial soln if and only if \mathbf{A} is row equivalent to the $n \times n$ identity matrix \mathbf{I} .



- ◆ **Every homogeneous linear system w/more variables (columns) than eqs (rows) has infinitely many sols.**
- ◆ **A non-homogeneous linear system w/more variables (columns) than eqs (rows) has infinitely many sols or NO sols.**

Exercises 3.3

What did we learn?

- ◆ Reduced Row Echelon Form-RREF
- ◆ Homogeneous Systems and their sols: $[\mathbf{A} | \vec{0}]$
- ◆ Non-Homogeneous Systems and their sols: $[\mathbf{A} | \vec{b}]$
- ◆ Square Coefficient Systems and their sols: $[\mathbf{A}^{n \times n} | \vec{0}]$

